M13
DIFFERENTIAL GEOMETRY

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Spring Semester 2024

Version 04 May 2024

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Contents

CHAPTER 1

Fundamental notions

WEEK 1

We start by introducing some notions that are fundamental for the study of curved spaces. Doing so will lead to a deeper understanding of some concepts from Linear Algebra and Analysis.

1.1 Points, vectors and the tangent space

Recall that we define $\mathbb{R}^n$ as ordered n-tuples $p = (x_1, \ldots, x_n)$ of scalars $x_i \in \mathbb{R}$, $1 \leqslant i \leqslant n$. We also consider *column vectors of length n with real entries*

$$\vec{v} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

We write $M_{m,n}(\mathbb{R})$ for the set of $(m \times n)$-matrices with real entries. A column vector of length n may be thought of as an $(n \times 1)$-matrix, hence we write $M_{n,1}(\mathbb{R})$ for the set of such column vectors. Clearly we have a bijective map

$$\Psi_n : \mathbb{R}^n \to M_{n,1}(\mathbb{R}), \quad (x_1, \ldots, x_n) \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

which writes the entries of an n-tuple into a column vector. Because of this map, we may avoid a distinction between $\mathbb{R}^n$ and $M_{n,1}(\mathbb{R})$ and pretend they are the same thing. This was done so in Linear Algebra. In geometry, it turns out to be useful to think of $\mathbb{R}^n$ and $M_{n,1}(\mathbb{R})$ as different sets. The elements of $\mathbb{R}^n$ are interpreted as *points* and will be denoted by $p, q, r \ldots$. The elements of $M_{n,1}(\mathbb{R})$ are interpreted as *vectors in* $\mathbb{R}^n$ *that are attached to the origin* $0_{\mathbb{R}^n} = (0, 0, \ldots, 0) \in \mathbb{R}^n$. They will be denoted by $\vec{u}, \vec{v}, \vec{w}, \ldots$.

Already in elementary geometry the situation occurs where we consider vectors in $\mathbb{R}^n$ that are *not* attached to the origin $0_{\mathbb{R}^n}$, but rather to some other point $p \in \mathbb{R}^n$. Think for instance of the normal vector of a plane in $\mathbb{R}^3$ not containing the origin $0_{\mathbb{R}^3}$.

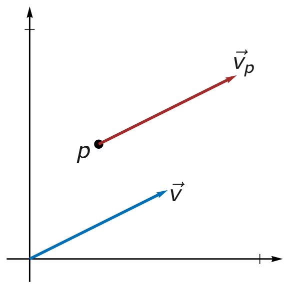

FIGURE 1.1. A vector $\vec{v}$ attached at the origin and at the point p.

In order to deal with vectors that are not attached to the origin, but to a point $p \in \mathbb{R}^n$, we introduce the so-called *tangent space of* $\mathbb{R}^n$ *at* p,

$$T_p\mathbb{R}^n = \{\vec{v}_p \,|\, \vec{v} \in M_{n,1}(\mathbb{R})\}\,.$$

The element $\vec{v}_p \in T_p\mathbb{R}^n$ is to be interpreted as attaching the vector $\vec{v} \in M_{n,1}(\mathbb{R})$ at the *basepoint* $p \in \mathbb{R}^n$. The elements of $T_p\mathbb{R}^n$ are called *tangent vectors with basepoint* p. Observe that for all $p \in \mathbb{R}^n$ the tangent space $T_p\mathbb{R}^n$ is a vector space over $\mathbb{R}$ when equipped with vector addition $+_{T_p\mathbb{R}^n} : T_p\mathbb{R}^n \times T_p\mathbb{R}^n \to T_p\mathbb{R}^n$ defined by the rule

$$\vec{v}_p +_{T_p\mathbb{R}^n} \vec{w}_p = (\vec{v} +_{M_{n,1}(\mathbb{R})} \vec{w})_p$$

for all $\vec{v}_p,\vec{w}_p \in T_p\mathbb{R}^n$ and scalar multiplication $\cdot_{T_p\mathbb{R}^n} : \mathbb{R} \times T_p\mathbb{R}^n \to T_p\mathbb{R}^n$ defined by the rule

$$s \cdot_{T_p\mathbb{R}^n} \vec{v}_p = (s \cdot_{M_{n,1}(\mathbb{R})} \vec{v})_p$$

for all $s \in \mathbb{R}$ and all $\vec{v}_p \in T_p\mathbb{R}^n$. Here $+_{M_{n,1}(\mathbb{R})}$ denotes usual component-wise addition of column vectors and $\cdot_{M_{n,1}(\mathbb{R})}$ denotes usual component-wise scalar multiplication of a column vector by a scalar. Clearly, for all $p \in \mathbb{R}^n$ we have a vector space isomorphism

$$T_p\mathbb{R}^n \to M_{n,1}(\mathbb{R}), \quad \vec{v}_p \mapsto \vec{v}$$

which simply "forgets" the *basepoint* $p \in \mathbb{R}^n$. We can thus think of $T_p\mathbb{R}^n$ as a copy of $M_{n,1}(\mathbb{R})$ attached to $p \in \mathbb{R}^n$. The union of all these copies of $\mathbb{R}^n$ is known as the *tangent bundle of* $\mathbb{R}^n$

$$T\mathbb{R}^n = \bigcup_{p\in\mathbb{R}^n} T_p\mathbb{R}^n = \bigcup_{p\in\mathbb{R}^n} \{\vec{v}_p \,|\, \vec{v} \in M_{n,1}(\mathbb{R})\}\,.$$

At this point the name *tangent space* is a bit confusing, since it is unclear to what $T_p\mathbb{R}^n$ is tangent to. This will be clarified later on. If $U \subset \mathbb{R}^n$ is an open subset, we define likewise

$$TU = \bigcup_{p\in U} T_p\mathbb{R}^n.$$

Observe that for each $p \in \mathbb{R}^n$ the tangent space $T_p\mathbb{R}^n$ is equipped with an ordered basis

$$\mathbf{e}_p^{(n)} = ((\vec{e}_1)_p, \ldots, (\vec{e}_n)_p),$$

where $\{\vec{e}_1, \ldots, \vec{e}_n\}$ denotes the standard basis of $M_{n,1}(\mathbb{R})$. For all $p \in \mathbb{R}^n$ we call $\mathbf{e}_p^{(n)}$ the ordered standard basis of $T_p\mathbb{R}^n$.

Whenever n is clear from the context we simply write $\mathbf{e}_p$ instead of $\mathbf{e}_p^{(n)}$.

Remark 1.1 Since $M_{1,1}(\mathbb{R})$ is one-dimensional, so is $T_t\mathbb{R}$ for all $t \in \mathbb{R}$ and the ordered standard basis of $T_t\mathbb{R}$ consists of a single vector which we denote by 1_t.

Recall that a pair $(V, \langle\cdot,\cdot\rangle)$ consisting of a vector space V over $\mathbb{R}$ and an inner product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ is called a *Euclidean space*.[1] We can turn each tangent space into a Euclidean space:

Definition 1.2 For all $p \in \mathbb{R}^n$, the *standard inner product on* $T_p\mathbb{R}^n$ is the unique inner product $\langle\cdot,\cdot\rangle_p$ for which $\mathbf{e}_p$ is an orthonormal basis, that is, we have

$$\langle(\vec{e}_i)_p, (\vec{e}_j)_p\rangle_p = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

[1] An inner product on an vector V over $\mathbb{R}$ is a positive definite symmetric bilinear form $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$.

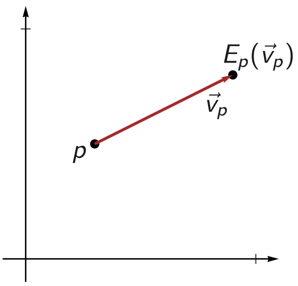

FIGURE 1.2. The endpoint $E_p(\vec{v}_p)$ of a tangent vector $\vec{v}_p$.

We will henceforth always assume that $T_p\mathbb{R}^n$ is equipped with $\langle\cdot,\cdot\rangle_p$. Whenever no confusion can arise about the point p at which $\langle\cdot,\cdot\rangle_p$ is computed, we will usually simply write $\langle\cdot,\cdot\rangle$.

Occasionally it is useful to turn a tangent vector $\vec{v}_p \in T_p\mathbb{R}^n$ into a point $q \in \mathbb{R}^n$. This is done by mapping a tangent vector $\vec{v}_p \in T_p\mathbb{R}^n$ to its "endpoint". More precisely, we define:

Definition 1.3 (Endpoint map) For all $p \in \mathbb{R}^n$ we define

$$E_p : T_p\mathbb{R}^n \to \mathbb{R}^n, \qquad \vec{v}_p \mapsto E_p(\vec{v}_p) = (x_1 + v_1, \ldots, x_n + v_n),$$

where $p = (x_1, \ldots, x_n)$ and

$$\vec{v}_p = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_p$$

1.2 Smooth maps, diffeomorphisms and the differential

We recall some facts from Analysis II, but now with a slightly more geometric perspective.

For $n \in \mathbb{N}$ we let $\{e_1, \ldots, e_n\}$ – here interpreted as points – denote the standard basis of $\mathbb{R}^n$, that is $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$ and so on. Let $U \subset \mathbb{R}^n$ be an open set and consider a map $f : U \to \mathbb{R}^m$. Recall that for all $p \in U$ and $1 \leqslant i \leqslant n$ we define the *partial derivative of f in the coordinate direction i* as

$$\partial_i f(p) = \lim_{h \to 0} \frac{1}{h} \left(f(p + he_i) - f(p) \right),$$

provided the limit exists. Recall also from Analysis II that the map $f : U \to \mathbb{R}^m$ is *continuously differentiable* if and only if[2] for all $1 \leqslant i \leqslant n$

(i) the partial derivative $\partial_i f(p)$ exists for all $p \in U$;
(ii) the map $\partial_i f : U \to \mathbb{R}^m$, $p \mapsto \partial_i f(p)$ is continuous.

Recursively, we can define higher derivatives. For $k \in \mathbb{N}$, $k \geqslant 2$ we call $f : U \to \mathbb{R}^m$ *k-times continuously differentiable* if $\partial_i f : U \to \mathbb{R}^m$ is $(k-1)$-times continuously differentiable for all $1 \leqslant i \leqslant n$. We write

$$C^k(U, \mathbb{R}^m) = \{f : U \to \mathbb{R}^m \mid f \text{ is } k\text{-times continuously differentiable}\}$$

and

[2] This is a theorem, not a definition!

Definition 1.4 We set

$$C^{\infty}(U,\mathbb{R}^m) = \bigcap_{k\in\mathbb{N}} C^{\kappa}(U,\mathbb{R}^m)$$

and call the elements of $C^{\infty}(U,\mathbb{R}^m)$ *smooth maps from U to $\mathbb{R}^m$*.

Throughout this module we will almost exclusively consider smooth maps.

Remark 1.5 (Smooth maps on non-open domains) It is useful to have a notion of smoothness for maps that are defined on some arbitrary subset $\mathcal{X} \subset \mathbb{R}^n$. A map $f : \mathcal{X} \to \mathbb{R}^m$ is called *smooth* if there exists an open subset $U \subset \mathbb{R}^n$ containing $\mathcal{X}$ and a smooth function $\hat{f} : U \to \mathbb{R}^m$ so that $\hat{f}(p) = f(p)$ for all $p \in \mathcal{X}$.

Definition 1.6 (Differential of a map) Given $U \subset \mathbb{R}^n$, let $f : U \to \mathbb{R}^m$ be smooth and write $f = (f_1, \dots, f_m)$ for real-valued functions $f_i : U \to \mathbb{R}$.

(i) The *differential of f at $p \in U$* is the unique linear map

$$f_*|_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$$

so that for all

$$\vec{v}_p = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_p \in T_p\mathbb{R}^n,$$

we have

$$f_*|_p(\vec{v}_p) = \vec{w}_{f(p)} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}_{f(p)}$$

with

$$\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} \partial_1 f_1(p) & \cdots & \partial_n f_1(p) \\ \vdots & \ddots & \vdots \\ \partial_1 f_m(p) & \cdots & \partial_n f_m(p) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}. \tag{1.1}$$

(ii) Recall that the $(n \times m)$-matrix on the right in (1.1) is called the *Jacobian matrix of f at p*. We denote it by $\mathbf{J}f(p)$.

(iii) For each $p \in U$ we obtain a linear map $f_*|_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$. It is useful to think of the family $\{f_*|p\}_{p\in U}$ of all such linear maps as a single map

$$f_* : TU \to T\mathbb{R}^m$$

defined by the rule

$$f_*(\vec{v}_p) = \vec{w}_{f(p)}, \qquad \text{where} \qquad \vec{w} = \mathbf{J}f(p)\vec{v}.$$

That is, for all $p \in U$, the restriction of f_* to $T_p\mathbb{R}^n \subset TU$ is given by $f_*|_p$. The map $f_* : TU \to T\mathbb{R}^m$ is called *the differential of f*.

Example 1.7 Consider the smooth map

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad p = (x,y) \mapsto f(p) = (x^2 - y^2, xy)$$

For the Jacobian we obtain

$$\mathbf{J}f(p) = \begin{pmatrix} 2x & -2y \\ y & x \end{pmatrix}$$

and hence for

$$\vec{v}_p = \begin{pmatrix} u \\ w \end{pmatrix}_{(x,y)}$$

we have

$$f_*(\vec{v}_p) = \begin{pmatrix} 2xu - 2yw \\ yu + xw \end{pmatrix}_{(x^2-y^2,xy)}.$$

Remark 1.8 (Matrices acting on points) Recall that $\Psi_n : \mathbb{R}^n \to M_{n,1}(\mathbb{R})$ is the map that turns a point into a column vector. We use Ψ_n to let an $(m \times n)$-matrix $\mathbf{A}$ act on points of $\mathbb{R}^n$ by the rule

$$\mathbf{A}p := \Psi_m^{-1}(\mathbf{A}\Psi_n(p))$$

for all $p \in \mathbb{R}^n$, where on the right hand side $\mathbf{A}$ acts on the column vector $\Psi_n(p)$ by matrix multiplication.

Example 1.9 Let $\mathbf{A} \in M_{m,n}(\mathbb{R})$, $b \in \mathbb{R}^m$ and consider the map

$$f_{\mathbf{A},b} : \mathbb{R}^n \to \mathbb{R}^m, \qquad p \mapsto \mathbf{A}p + b.$$

Then we have

$$(f_{\mathbf{A},b})_*(\vec{v}_p) = (\mathbf{A}\vec{v})_{\mathbf{A}p+b}.$$

for all $p \in \mathbb{R}^n$ and $\vec{v}_p \in T_p\mathbb{R}^n$.

Definition 1.10 (Euclidean motion) A map

$$f_{\mathbf{R},q} : \mathbb{R}^n \to \mathbb{R}^n, \qquad p \mapsto \mathbf{R}p + q$$

for some point $q \in \mathbb{R}^n$ and orthogonal matrix $\mathbf{R} \in \mathrm{O}(n)$ is called a *Euclidean motion*.

Example 1.11 For $n = 2$, $q = (y_1, y_2)$ and

$$\mathbf{R} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \qquad \alpha \in \mathbb{R},$$

we have

$$f_{\mathbf{R},q}(p) = (\cos(\alpha)x_1 - \sin(\alpha)x_2 + y_1, \sin(\alpha)x_1 + \cos(\alpha)x_2 + y_2)$$

where we write $p = (x_1, x_2)$.

Example 1.12 Consider a Euclidean motion $f_{\mathbf{R},q} : \mathbb{R}^n \to \mathbb{R}^n$, then

$$(f_{\mathbf{R},q})_*(\vec{v}_p) = (\mathbf{R}\vec{v})_{\mathbf{R}p+q}.$$

Notice that this implies that

$$\langle (f_{\mathbf{R},q})_*(\vec{v}_p), (f_{\mathbf{R},q})_*(\vec{w}_p) \rangle = \langle \vec{v}_p, \vec{w}_p \rangle$$

for all $p \in \mathbb{R}^n$ and all $\vec{v}_p, \vec{w}_p \in T_p\mathbb{R}^n$.

Remark 1.13 Let $U \subset \mathbb{R}$ be open and $f : U \to \mathbb{R}$ a smooth function. We have the usual derivative from Analysis I

$$f' : U \to \mathbb{R}, \qquad t \mapsto f'(t) = \frac{\mathrm{d}f}{\mathrm{d}t}(t) = \lim_{h\to 0} \frac{1}{h}(f(t+h) - f(t)).$$

We also have the differential in the sense of Definition 1.6 which is a map $f_* : TU \to T\mathbb{R}$. Now notice that for all $t \in U$ we have

$$f_*(1_t) = f'(t)1_{f(t)}. \tag{1.2}$$

Recommendation: Pause here and think about (1.2) until you understand it.

Diffeomorphisms are smooths maps that are bijective and admit a smooth inverse:

Definition 1.14 (Diffeomorphism) Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets and $f : U \to V$ a smooth map. If f is bijective and $f^{-1} : V \to U$ is smooth as well, then $f : U \to V$ is called a *diffeomorphism*.

Recall from Analysis that if $f : U \to V$ is a diffeomorphism, then $n = m$ and moreover, for all $p \in U$ the linear map $f_*|_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^n$ is invertible.

If $f : U \to \mathbb{R}^m$ is a smooth and injective map, we say f *is a diffeomorphism onto its image*, provided the inverse map $f^{-1} : \mathrm{Im}(f) \to U$ is smooth as well. Here as usual we define

$$\mathrm{Im}(f) = f(U) = \{q \in \mathbb{R}^m | q = f(p),\ p \in U\}.$$

From the chain rule in Analysis II we conclude:

Proposition 1.15 (Chain rule) *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets and $f : U \to \mathbb{R}^m$ and $g : V \to \mathbb{R}^k$ be smooth maps with $f(U) \subset V$. Then $g \circ f : U \to \mathbb{R}^k$ is smooth and for all $p \in U$ we have*

$$(g \circ f)_*|_p = g_*|_{f(p)} \circ f_*|_p. \tag{1.3}$$

That is, the differential of the composition $g \circ f$ at p is given by the composition of the linear map $f_*|_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$ and the linear map $g_*|_{f(p)} : T_{f(p)}\mathbb{R}^m \to T_{g(f(p))}\mathbb{R}^k$.

Remark 1.16 (Sums and products of smooth maps) The chain rule tells us that compositions of smooth maps are smooth, so are sums and products. More precisely:

(i) If $f, g : U \to \mathbb{R}^m$ are smooth, then so is $f +_{C^\infty(U,\mathbb{R}^m)} g : U \to \mathbb{R}^m$, where

$$(f +_{C^\infty(U,\mathbb{R}^m)} g)(p) = f(p) +_{\mathbb{R}^m} g(p)$$

for all $p \in U$.

(ii) If $f, g : U \to \mathbb{R}$ are smooth, then so is $f \cdot_{C^\infty(U,\mathbb{R})} g : U \to \mathbb{R}$, where

$$(f \cdot_{C^\infty(U,\mathbb{R})} g)(p) = f(p) \cdot_{\mathbb{R}} g(p)$$

for all $p \in U$.

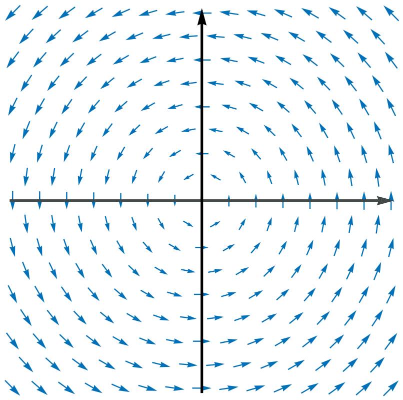

FIGURE 1.3. A visualisation of the vector field $p = (x_1, x_2) \mapsto \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}_p$.

1.3 Vector fields and the gradient

Recall that for all $p \in \mathbb{R}^n$ the tangent space $T_p\mathbb{R}^n$ is equipped with a basis given by attaching the standard basis $\{\vec{e}_1, \dots, \vec{e}_n\}$ of $M_{n,1}(\mathbb{R})$ at p. We may think of attaching the i-th standard basis vector at p as a map from $\mathbb{R}^n$ to $T\mathbb{R}^n$. That is, we define

$$\frac{\partial}{\partial x_i} : \mathbb{R}^n \to T\mathbb{R}^n, \qquad p \mapsto \frac{\partial}{\partial x_i}(p) = (\vec{e}_i)_p .$$

The mappings $\frac{\partial}{\partial x_i}$ are examples of vector fields:

Definition 1.17 (Vector field) A *vector field* on some open subset $U \subset \mathbb{R}^n$ is a map $X : U \to T\mathbb{R}^n$ so that $X(p) \in T_p\mathbb{R}^n$ for all $p \in U$. For a vector field $X : U \to T\mathbb{R}^n$ there exists unique functions $X_i : U \to \mathbb{R}$, $1 \leqslant i \leqslant n$, so that

$$X(p) = \sum_{i=1}^{n} X_i(p) \frac{\partial}{\partial x_i}(p)$$

for all $p \in U$. The vector field is called *smooth* if the functions X_i are smooth for all $1 \leqslant i \leqslant n$.

Remark 1.18 By definition, $\frac{\partial}{\partial x_i}$ is a map from $\mathbb{R}^n \to T\mathbb{R}^n$. The notation $\frac{\partial}{\partial x_i}$ might seem strange for a map, it will be motivated below.

A vector field simply attaches a tangent vector v_p to every point p of its domain of definition. Vector fields appear naturally in physics. For instance, an electromagnetic field is an example of a vector field. Likewise, in the classical Newtonian theory of gravity, the gravitational field is an example of a vector field.

Example 1.19 Write $p = (x_1, x_2)$ for an element of $\mathbb{R}^2$, then

$$X : \mathbb{R}^2 \to T\mathbb{R}^2, \qquad p = (x_1, x_2) \mapsto \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}_p$$

is a smooth vector field on $\mathbb{R}^2$.

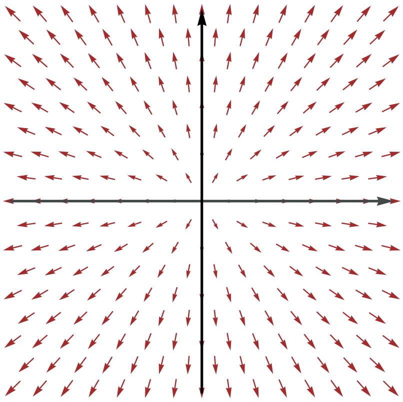

FIGURE 1.4. A visualisation of the gradient of the function $p = (x_1, x_2) \mapsto (x_1)^2 + (x_2)^2$.

Every smooth function gives rise to a vector field:

Definition 1.20 (Gradient) Let $U \subset \mathbb{R}^n$ and $f : U \to \mathbb{R}$ be a smooth function. Then the so-called *gradient of f* defined by

$$\operatorname{grad} f : U \to T\mathbb{R}^n, \qquad p \mapsto \begin{pmatrix} \partial_1 f(p) \\ \vdots \\ \partial_n f(p) \end{pmatrix}_p$$

is a smooth vector field on U.

Example 1.21 Consider the smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by the rule

$$f(p) = (x_1)^2 + (x_2)^2,$$

where we write $p = (x_1, x_2)$. Then we have

$$\operatorname{grad} f(p) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}_p .$$

1.4 The cotangent space and the exterior derivative

Recall from Linear Algebra II that if V is a vector space over $\mathbb{R}$, then its *dual vector space* V^* consists of the linear maps $f : V \to \mathbb{R}$ with vector addition defined by the rule

$$(f +_{V^*} g)(v) = f(v) +_{\mathbb{R}} g(v)$$

for all $f, g \in V^*$ and all $v \in V$ and scalar multiplication defined by the rule

$$(s \cdot_{V^*} f) = s \cdot_{\mathbb{R}} f(v)$$

for all $f \in V^*$, all $s \in \mathbb{R}$ and all $v \in V$.

We can think of row vectors of length n with real entries, that is the elements of $M_{1,n}(\mathbb{R})$, as elements of the dual vector space of the vector space $M_{n,1}(\mathbb{R})$ of column vectors of length n with real entries. This is done by interpreting $\vec{\nu} \in M_{1,n}(\mathbb{R})$ as a linear map

$M_{n,1}(\mathbb{R}) \to \mathbb{R}$ given by

$$\vec{\nu}(\vec{v}) = \vec{\nu}\vec{v} \in \mathbb{R},$$

where on the right hand side we use matrix multiplication of the row vector $\vec{\nu}$ and the column vector $\vec{v}$. In doing so, the standard basis of $M_{1,n}(\mathbb{R})$ given by $\{\vec{\varepsilon}_1, \dots, \vec{\varepsilon}_n\}$ can be interpreted as as basis of $(M_{n,1}(\mathbb{R}))^*$. Here $\vec{\varepsilon}_i$ denotes the row vector of length n whose i-th entry is 1 with all other entries 0.

Let $n \in \mathbb{N}$ and $p \in \mathbb{R}^n$. The dual vector space of the tangent space $T_p\mathbb{R}^n$ is called the *cotangent space* at $p \in M$ and denoted by $T_p^*\mathbb{R}^n := (T_p\mathbb{R}^n)^*$. We write an element of $T_p^*\mathbb{R}^n$ as $\vec{\nu}_p$ with $p \in \mathbb{R}^n$ and $\vec{\nu} \in M_{1,n}(\mathbb{R})$. Hence we have

$$T_p^*\mathbb{R}^n = \{\vec{\nu}_p \,|\, \vec{\nu} \in M_{1,n}(\mathbb{R})\}.$$

The elements of $T_p^*\mathbb{R}^n$ are called *cotangent vectors with basepoint* p. The union of all cotangent spaces is the so-called *cotangent bundle*

$$T^*\mathbb{R}^n = \bigcup_{p \in \mathbb{R}^n} T_p^*\mathbb{R}^n = \bigcup_{p \in \mathbb{R}^n} \{\vec{\nu}_p \,|\, \vec{\nu} \in M_{1,n}(\mathbb{R})\}.$$

As in the case of the tangent space, each cotangent space $T_p^*\mathbb{R}^n$ is equipped with an ordered basis $\varepsilon_p^{(n)} = ((\vec{\varepsilon}_1)_p, \dots, (\vec{\varepsilon}_n)_p)$.

Definition 1.22 (Exterior derivative) Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ a smooth function.

(i) The *exterior derivative of f at $p \in U$* is the unique cotangent vector $\mathrm{d}f|_p \in T_p^*\mathbb{R}^n$ so that

$$\mathrm{d}f|_p = \partial_1 f(p)(\vec{\varepsilon}_1)_p + \cdots + \partial_n f(p)(\vec{\varepsilon}_n)_p. \tag{1.4}$$

(ii) As in the case of the differential, there exists a unique map

$$\mathrm{d}f : TU \to \mathbb{R}$$

so that for all $p \in U$, the restriction of $\mathrm{d}f$ to $T_p\mathbb{R}^n \subset TU$ is given by $\mathrm{d}f|_p$. The map $\mathrm{d}f : TU \to \mathbb{R}$ is called the *exterior derivative of f*.

Remark 1.23 (Exterior derivative vs the gradient) Notice that for a smooth function $f : U \to \mathbb{R}$ we have for all $\vec{v}_p \in TU$

$$\langle \operatorname{grad} f(p), \vec{v}_p \rangle = \mathrm{d}f(\vec{v}_p).$$

Remark 1.24 (Exterior derivative vs the differential) Notice that the differential and the exterior derivative are not the same thing! The differential is defined for smooth maps $f : U \to \mathbb{R}^m$, whereas the exterior derivative is only defined for smooth functions, that is, smooth maps $f : U \to \mathbb{R}$ taking values in the real numbers. For a function $f : U \to \mathbb{R}$ the two notions are however closely related. The differential of f is a map

$$f_* : TU \to T\mathbb{R}$$

and the exterior derivative of f at p is a map

$$\mathrm{d}f : TU \to \mathbb{R}.$$

Recall that we have a natural basis of $T_{f(p)}\mathbb{R}$ consisting of the vector $1_{f(p)}$ and with respect to this basis we have for all $\vec{v}_p \in T_p\mathbb{R}^n$

$$f_*(\vec{v}_p) = \mathrm{d}f(\vec{v}_p)1_{f(p)}.$$

Remark 1.25 (Standard abuse of notation) It is customary in the literature to use the letter x both for an unspecified element in $\mathbb{R}^n$, as well as for the identity map on $\mathbb{R}^n$

$$x = \mathrm{Id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n, \quad p \mapsto x(p) = p.$$

This can – and usually does – lead to confusion. Unfortunately this is well established notation used in almost all books about differential geometry. We will therefore adopt it as well.
For $1 \leqslant i \leqslant n$ the projection onto the i-th entry of a point $p \in \mathbb{R}^n$ is denoted by x_i

$$x_i : \mathbb{R}^n \to \mathbb{R}, \quad p = (p_1, \dots, p_n) \mapsto x_i(p) = p_i.$$

Notice that for $1 \leqslant i, j \leqslant n$

$$\partial_i x_j(p) = \delta_{ij}$$

and hence

$$\mathrm{d}x_i|_p = (\vec{\varepsilon}_i)_p. \tag{1.5}$$

Combining (1.4) and (1.5) we obtain for a smooth function $f : U \to \mathbb{R}$ and all $p \in U$

$$\mathrm{d}f|_p = \partial_1 f(p)\mathrm{d}x_1|_p + \cdots + \partial_n f(p)\mathrm{d}x_n|_p.$$

When omitting the basepoint p, we get

$$\mathrm{d}f = \partial_1 f \mathrm{d}x_1 + \cdots + \partial_n f \mathrm{d}x_n.$$

If $f : I \to \mathbb{R}$ is a smooth function on an interval, the previous equations become

$$\mathrm{d}f|_u = f'(u)\mathrm{d}t|_u \quad \text{and} \quad \mathrm{d}f = f'\mathrm{d}t,$$

where here $u \in I$ and t denotes the identity map on $\mathbb{R}$.

Example 1.26 (Exterior derivative) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the smooth function defined by the rule

$$f(x_1, x_2) = \mathrm{e}^{2x_1}\sin(x_2).$$

Then we obtain for the exterior derivative

$$\mathrm{d}f = 2\mathrm{e}^{2x_1}\sin(x_2)\mathrm{d}x_1 + \mathrm{e}^{2x_1}\cos(x_2)\mathrm{d}x_2.$$

Definition 1.27 (Directional derivative) Let $U \subset \mathbb{R}^m$ be open and $f : U \to \mathbb{R}$ a smooth function.

(i) For a tangent vector $\vec{v}_p \in T_p\mathbb{R}^n \subset TU$, we define the *directional derivative of f at p in the direction $\vec{v}_p$* by $\mathrm{d}f(\vec{v}_p)$.

(ii) Given a smooth vector field $X : U \to T\mathbb{R}^n$, we obtain a smooth function

$$X(f) : U \to \mathbb{R}, \quad p \mapsto X(f)(p) := \mathrm{d}f(X(p))$$

whose value at $p \in U$ is given by the directional derivative of f at p in the direction $X(p) \in T_p\mathbb{R}^n$.

(iii) When $X = \frac{\partial}{\partial x_i}$ for $1 \leqslant i \leqslant n$ it is customary to write

$$\frac{\partial f}{\partial x_i} := \frac{\partial}{\partial x_i}(f).$$

Notice that

$$\frac{\partial f}{\partial x_i} = \partial_i f.$$

for all $1 \leqslant i \leqslant n$.

(iv) Writing $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$ for smooth functions $X_i : U \to \mathbb{R}$, we obtain

$$X(f) = \sum_{i=1}^n X_i \frac{\partial f}{\partial x_i},$$

where we use that $\mathrm{d}f|_p : T_p\mathbb{R}^n \to \mathbb{R}$ is linear.

Example 1.28 For the vector field X defined in Example 1.19 and the function f defined in Example 1.26 we thus obtain

$$X(f) = -x_2 2\mathrm{e}^{2x_1} \sin(x_2) + x_1 \mathrm{e}^{2x_1} \cos(x_2).$$

CHAPTER 2

Curves

WEEK 2

Curves are among the simplest geometric objects we can study, but they already have non-trivial properties.

2.1 Definitions and examples

Definition 2.1 (Curve) Let $m \in \mathbb{N}$ and $I \subset \mathbb{R}$ be an interval. A curve in $\mathbb{R}^m$ is a continuous map $\gamma = (\gamma_i)_{1 \leqslant i \leqslant m} : I \to \mathbb{R}^m$. The curve γ is called *smooth* if $\gamma : I \to \mathbb{R}^m$ is a smooth map.

Definition 2.2 (Velocity vector) Let $\gamma = (\gamma_i)_{1 \leqslant i \leqslant m} : I \to \mathbb{R}^m$ be a smooth curve.

(i) We define the velocity vector of γ at time $t \in I$ by

$$\dot{\gamma}(t) = \gamma_*(1_t).$$

Notice that the velocity vector of γ at time $t \in I$ is an element of the tangent space $T_{\gamma(t)}\mathbb{R}^m$ at $\gamma(t)$.

(ii) The map

$$\dot{\gamma} : I \to T\mathbb{R}^m, \qquad t \mapsto \dot{\gamma}(t)$$

is called the *velocity vector field along* γ.

(iii) A smooth curve γ satisfying $\dot{\gamma}(t) \neq 0_{T_{\gamma(t)}\mathbb{R}^m}$ for all $t \in I$ is called an *immersed curve*.

Definition 2.3 (Speed and length of a curve) Let $\gamma : I \to \mathbb{R}^m$ be a smooth curve.

(i) The *speed* of γ at time $t \in I$ is defined as

$$\|\dot{\gamma}(t)\| = \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}.$$

(ii) If $I = [a, b]$ for real numbers $a < b$, we define the length of γ as

$$\ell(c) = \int_a^b \|\dot{\gamma}(t)\| \mathrm{d}t.$$

Remark 2.4 Let $\gamma : I \to \mathbb{R}^m$ be a smooth curve. Then its Jacobian is

$$\mathbf{J}\gamma(t) = \begin{pmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_m'(t) \end{pmatrix}.$$

In particular, we obtain

$$\dot{\gamma}(t) = \gamma_*(1_t) = \vec{w}_{\gamma(t)} = \begin{pmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_m'(t) \end{pmatrix}_{\gamma(t)}$$

where $\vec{w} = \mathbf{J}\gamma(t)1$. The velocity vector $\dot{\gamma}(t)$ at time t is thus simply obtained by computing the usual time derivatives $\gamma_i'(t)$ of the components γ_i of γ at time t and attaching the resulting vector at $\gamma(t)$.

Example 2.5 (Unit circle) The curve

$$\gamma : [0, 2\pi] \to \mathbb{R}^2, \qquad t \mapsto (\cos(t), \sin(t))$$

is smooth and its image $\gamma([0, 2\pi])$ consists of the circle of radius 1 centred at $(0, 0)$. The curve γ has velocity vector

$$\dot{\gamma}(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}_{\gamma(t)}$$

and speed

$$\|\dot{\gamma}(t)\| = \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle} = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1$$

at time $t \in [0, 2\pi]$, respectively. Therefore, the unit circle has length

$$\ell(c) = \int_0^{2\pi} \|\dot{\gamma}(t)\| \mathrm{d}t = \int_0^{2\pi} 1 \mathrm{d}t = 2\pi.$$

Example 2.6 (Non-immersed curve) The curve

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad t \mapsto (t^3, t^2)$$

is smooth with velocity vector

$$\dot{\gamma}(t) = \begin{pmatrix} 3t^2 \\ 2t \end{pmatrix}_{\gamma(t)}$$

and speed

$$\|\dot{\gamma}(t)\| = \sqrt{9t^4 + 4t^2}.$$

Since $\gamma'(0) = 0_{\mathbb{R}^2}$ it is not an immersion.

Example 2.7 (Helix) The curve

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \qquad t \mapsto (\cos(t), \sin(t), t)$$

is smooth and its image $\gamma(\mathbb{R})$ consists of a helix.

Example 2.8 (Figure-eight curve) The curve

$$\gamma : [0, 2\pi] \to \mathbb{R}^3, \qquad t \mapsto ((2 + \cos(2t))\cos(3t), (2 + \cos(2t))\sin(3t), \sin(4t))$$

is smooth and its image $\gamma([0, 2\pi])$ consists of a figure-eight knot.

Remark 2.9 (Curves and the differential) Given $U \subset \mathbb{R}$, let $f : U \to \mathbb{R}^m$ be a smooth map. For $p \in U$ and $\vec{v}_p \in T_p\mathbb{R}^n$ we would like to interpret the tangent vector $f_*(\vec{v}_p) \in T_{f(p)}\mathbb{R}^m$. For $\varepsilon > 0$ consider a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to U$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = \vec{v}_p$. For instance the curve

$$\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n, \qquad t \mapsto \gamma(t) = E_p(t\vec{v}_p)$$

satisfies $\gamma(0) = p$ and $\dot{\gamma}(0) = \vec{v}_p$. Recall that $E_p : T_p\mathbb{R}^n \to \mathbb{R}^n$ denotes the endpoint map from Definition 1.3. The composition of f and γ is then a smooth curve $\xi = f \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^m$ satisfying $\xi(0) = f(\gamma(0)) = f(p)$ and velocity vector

$$\dot{\xi}(0) = (f \circ \gamma)_*(1_0) = f_*(\gamma_*(1_0)) = f_*(\dot{\gamma}(0)) = f_*(\vec{v}_p),$$

where the second equality sign uses the chain rule (1.3). The image of $\vec{v}_p$ under f_* can thus be interpreted as the velocity vector at 0 of the curve $\xi = f \circ \gamma$.

If $\gamma : I \to \mathbb{R}^m$ is a smooth curve, the map $\dot{\gamma} : I \to T\mathbb{R}^m$, $t \mapsto \dot{\gamma}(t)$ assigns a tangent vector at $\gamma(t)$ to every time $t \in I$. This is an example of a *vector field along a curve*.

Definition 2.10 (Vector field along a curve) Let $\gamma : I \to \mathbb{R}^m$ be a curve. A map

$$X : I \to T\mathbb{R}^m, \qquad t \mapsto X(t)$$

is called a *vector field along* γ if $X(t) \in T_{\gamma(t)}\mathbb{R}^m$ for all $t \in I$. There exist unique functions $X_i : I \to \mathbb{R}$, $1 \leqslant i \leqslant m$ so that

$$X(t) = \sum_{i=1}^{m} X_i(t) \frac{\partial}{\partial x_i}(\gamma(t)).$$

The vector field X along γ is called *smooth* if the functions $X_i : I \to \mathbb{R}$ are smooth for all $1 \leqslant i \leqslant m$.

2.2 Unit speed curves

Definition 2.11 (Parameter on an interval) A smooth parameter on an interval I is a smooth injective map $\varphi : I \to \mathbb{R}$ which is a diffeomorphism onto its image $J = \varphi(I)$. The inverse map $\varphi^{-1} : J \to I$ is called a *parametrisation of* I.

Example 2.12 The sigmoid function

$$\varphi : \mathbb{R} \to (0, 1), \qquad t \mapsto \frac{1}{1 + e^{-t}}$$

is a smooth parameter on $\mathbb{R}$.

Example 2.13

(i) The tangent function

$$\varphi : (-\pi/2, \pi/2) \to \mathbb{R}, \qquad t \mapsto \tan(t)$$

is a smooth parameter on $(-\pi/2, \pi/2)$.

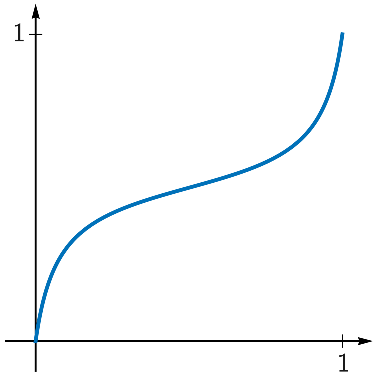

FIGURE 2.1. The graph of the parameter φ from (ii) for the choice $\delta = \pi/2 - 1/5$.

(ii) For $\delta \in (0, \pi/2)$ we can use the tangent function to define a smooth parameter

$$\varphi : [0,1] \to [0,1], \qquad t \mapsto \frac{\tan(-\delta + 2t\delta) - \tan(-\delta)}{2\tan(\delta)}$$

with

$$\varphi^{-1} : [0,1] \to [0,1], \qquad s \mapsto \frac{1}{2}\left(1 - \frac{\arctan(\beta - 2\beta s)}{\arctan(\beta)}\right),$$

where $\beta = \tan(\delta)$.

Example 2.14 Recall from Linear Algebra that a *linear coordinate system* on a finite dimensional vector space V over $\mathbb{R}$ is a linear injective map $\varphi : V \to \mathbb{R}^n$, where $n = \dim(V)$. A linear coordinate system on $\mathbb{R}$ –thought of as a 1-dimensional vector space over $\mathbb{R}$ – is thus also a smooth parameter on $\mathbb{R}$.

Remark 2.15 In light of Example 2.14 we can think of a smooth parameter on some interval I as a coordinate system on I which is allowed to be non-linear. We may think of this as a notion of non-linear time – see the animation below.

It often simplifies computations if a curve $\gamma : [a, b] \to \mathbb{R}^n$ has constant *unit speed*, that is, we have $\|\dot\gamma(t)\| = 1$ for all $t \in [a, b]$. Given an immersed curve $\gamma : [a, b] \to \mathbb{R}^n$ of length L, we may thus ask whether there exists a smooth unit speed curve $\xi : [0, L] \to \mathbb{R}^n$ such that

(i) $\xi([0, L]) = \gamma([a, b])$;
(ii) $\xi(0) = \gamma(a)$;
(iii) $\xi(L) = \gamma(b)$.

Intuitively, these conditions mean that ξ travels along the same *route* as γ (condition (i)), while starting at the same point $\xi(0) = \gamma(a)$ (condition (ii)) and ending at the same point $\xi(L) = \gamma(b)$ (condition (iii)). In order to find ξ we thus have to find a suitable *schedule* of how to move along γ. This leads to the notion of a reparametrisation.

Definition 2.16 (Reparametrisation of a curve) Let $\gamma : I \to \mathbb{R}^n$ be a smooth curve. A *reparametrisation* of γ is a smooth curve $\xi = \gamma \circ \varphi^{-1} : J \to \mathbb{R}^n$, where $\varphi : I \to J$ is a smooth parameter on I.

Proposition 2.17 (Existence of a unit-speed parametrisation) *Let $\gamma : [a, b] \to \mathbb{R}^n$ be a smooth immersed curve of length L. Then there exists a smooth parameter $s : [a, b] \to [0, L]$ so that the reparametrisation*

$$\xi = \gamma \circ s^{-1} : [0, L] \to \mathbb{R}^n$$

of γ is a unit speed curve.

Proof Consider the map

$$s : [a, b] \to \mathbb{R}, \quad t \mapsto \int_a^t \|\dot{\gamma}(u)\| \mathrm{d}u.$$

Clearly, we have $s(a) = 0$ and $s(b) = L = \int_a^b \|\dot{\gamma}(u)\| \mathrm{d}u$. By the fundamental theorem of calculus, the map s is differentiable and we have

$$s'(t) = \|\dot{\gamma}(t)\|. \tag{2.1}$$

Since γ is an immersed curve we have $\dot{\gamma}(t) \neq 0_{T_{\gamma(t)}\mathbb{R}^n}$ for all $t \in [a, b]$ and hence $s'(t) = \|\dot{\gamma}(t)\| > 0$ for all $t \in [a, b]$. Results from Analysis I imply that s is strictly increasing and thus a bijective map onto its image $[0, L]$. Moreover $s^{-1} : [0, L] \to [a, b]$ is differentiable for all $u \in [0, L]$ and we have

$$(s^{-1})'(u) = \frac{1}{s'(t)}, \tag{2.2}$$

where $s(t) = u$. In particular, $s : [a, b] \to [0, L]$ is smooth and moreover a diffeomorphism. It remains to show that $\xi = \gamma \circ s^{-1}$ is a unit speed curve. Using the chain rule we compute for all $u \in [0, L]$

$$\begin{aligned} \langle \dot{\xi}(u), \dot{\xi}(u) \rangle_{\xi(u)} &= \langle (\gamma \circ s^{-1})_* (1_u), (\gamma \circ s^{-1})_* (1_u) \rangle_{\gamma(s^{-1}(u))} \\ &= \langle \gamma_*(s_*^{-1}(1_u)), \gamma_*(s_*^{-1}(1_u)) \rangle_{\gamma(s^{-1}(u))}. \end{aligned}$$

Since by (1.2) we have

$$s_*^{-1}(1_u) = (s^{-1})'(u) 1_{s^{-1}(u)},$$

we obtain

$$\begin{aligned} \langle \dot{\xi}(u), \dot{\xi}(u) \rangle_{\xi(u)} &= \left[(s^{-1})'(u)\right]^2 \langle \gamma_*(1_{s^{-1}(u)}), \gamma_*(1_{s^{-1}(u)}) \rangle_{\gamma(s^{-1}(u))} \\ &= \left[(s^{-1})'(u)\right]^2 \|\dot{\gamma}(s^{-1}(u))\|^2 \\ &= \frac{\|\dot{\gamma}(s^{-1}(u))\|^2}{[s'(s^{-1}(u))]^2} = \frac{\|\dot{\gamma}(s^{-1}(u))\|^2}{\|\dot{\gamma}(s^{-1}(u))\|^2} = 1, \end{aligned}$$

where we use the linearity of $\gamma_*|_{s^{-1}(u)}$, the bilinearity of $\langle \cdot, \cdot \rangle_{\gamma(s^{-1}(u))}$ as well as (2.1) and (2.2). □

Remark 2.18 (Arc length)

(i) An *arc* is any smooth curve joining two points.

(ii) The parameter $s : [a, b] \to [0, L]$ associated to a smooth curve $\gamma : [a, b] \to \mathbb{R}^n$ of length L is called the *arc length parameter* of the curve, since $s(t)$ is the length of the arc connecting $\gamma(a)$ and $\gamma(t)$.

(iii) The curve $\gamma \circ s^{-1} : [0, L] \to \mathbb{R}^n$ is called the *parametrisation by arc length* of γ. For the curve $\xi = \gamma \circ s^{-1}$ the *travel time* $u \in [0, L]$ agrees with the distance travelled along ξ from $\xi(0)$ to $\xi(u)$.

Example 2.19 (Logarithmic spiral) For $b > 0$ and $A > 0$ and $t_0 \in \mathbb{R}$ consider the curve

$$\gamma : [t_0, \infty) \to \mathbb{R}^2, \qquad t \mapsto (Ae^{bt}\cos(t), Ae^{bt}\sin(t)).$$

It has speed

$$\|\dot\gamma(t)\| = Ae^{bt}\sqrt{b^2+1}$$

for all $t \in [t_0, \infty)$ and its arc length parameter is given by

$$s(t) = A\sqrt{b^2+1}\int_{t_0}^{t} e^{bu}\mathrm{d}u = A\frac{\sqrt{b^2+1}}{b}\left(e^{bt} - e^{bt_0}\right).$$

for all $t \in [t_0, \infty)$. Notice that

$$\lim_{t_0 \to -\infty}\int_{t_0}^{t}\|\dot\gamma(u)\|\mathrm{d}u = A\frac{\sqrt{b^2+1}}{b}e^{bt},$$

so that the arc length parameter is also well defined when we think of γ being defined on all of $\mathbb{R}$.

2.3 Curves in the plane

2.3.1 Curvature of a plane curve

Let $\gamma : I \to \mathbb{R}^m$ be a smooth curve with unit speed. Then we have for all $t \in I$

$$1 = \langle\dot\gamma(t), \dot\gamma(t)\rangle = \sum_{i=1}^{m}\left(\gamma_i'(t)\right)^2.$$

Taking the derivative with respect to t, we obtain

$$0 = \sum_{i=1}^{m} 2\gamma_i'(t)\gamma_i''(t). \tag{2.3}$$

Definition 2.20 (Acceleration vector) Let $\gamma : I \to \mathbb{R}^m$ be a smooth curve with velocity vector field $\dot\gamma : I \to T\mathbb{R}^m$ along γ. Then the *acceleration* vector field along γ is defined by

$$\ddot\gamma : I \to T\mathbb{R}^m, \qquad t \mapsto \ddot\gamma(t) = \begin{pmatrix}\gamma_1''(t)\\ \vdots \\ \gamma_m''(t)\end{pmatrix}_{\gamma(t)}$$

We call $\ddot\gamma(t)$ the *acceleration vector* of γ at time $t \in I$.

Using the notion of the acceleration vector, (2.3) can be written as

$$0 = 2\langle\dot\gamma(t), \ddot\gamma(t)\rangle.$$

We conclude that for a unit speed curve the velocity vector $\dot\gamma(t)$ and the acceleration vector $\ddot\gamma(t)$ are orthogonal for all $t \in I$.

Now we consider a smooth unit speed curve $\gamma : I \to \mathbb{R}^2$ in the plane $\mathbb{R}^2$. For $p \in \mathbb{R}^2$ let $J_p : T_p\mathbb{R}^2 \to T_p\mathbb{R}^2$ denote the counter-clockwise rotation by $\pi/2$ around $\vec{0}_p \in T_p\mathbb{R}^2$.[1] More precisely, $J_p : T_p\mathbb{R}^2 \to T_p\mathbb{R}^2$ is the unique linear map satisfying

$$J_p((\vec{e}_1)_p) = (\vec{e}_2)_p \quad \text{and} \quad J_p((\vec{e}_2)_p) = -(\vec{e}_1)_p. \tag{2.4}$$

Whenever p is clear from the context we will write J instead of J_p. Since $\dot{\gamma}(t)$ and $\ddot{\gamma}(t)$ are orthogonal for all $t \in I$, there exists a unique smooth function $\kappa : I \to \mathbb{R}$ so that

$$\ddot{\gamma}(t) = \kappa(t)J(\dot{\gamma}(t)) \tag{2.5}$$

The function $\kappa : I \to \mathbb{R}$ is called the *signed curvature* of γ. Notice that since $\|\dot{\gamma}(t)\| = 1$ for all $t \in I$ we also have $\|J(\dot{\gamma}(t))\| = 1$ and hence

$$\langle \ddot{\gamma}(t), J(\dot{\gamma}(t))\rangle = \langle \kappa(t)J(\dot{\gamma}(t)), J(\dot{\gamma}(t))\rangle = \kappa(t)\langle J(\dot{\gamma}(t)), J(\dot{\gamma}(t))\rangle = \kappa(t)$$

so that we obtain the identity

$$\kappa(t) = \langle \ddot{\gamma}(t), J(\dot{\gamma}(t))\rangle \tag{2.6}$$

for all $t \in I$.

Example 2.21 (Circle of radius r) For $r > 0$ consider the unit speed curve

$$\gamma : [0, 2\pi r] \to \mathbb{R}^2, \qquad t \mapsto (r\cos(t/r), r\sin(t/r)).$$

The image $\gamma([0, 2\pi r])$ consists of a circle of radius r centred at $0_{\mathbb{R}^2}$. We compute

$$\dot{\gamma}(t) = \begin{pmatrix} -\sin(t/r) \\ \cos(t/r) \end{pmatrix}_{\gamma(t)}$$

and

$$\ddot{\gamma}(t) = \frac{1}{r}\begin{pmatrix} -\cos(t/r) \\ -\sin(t/r) \end{pmatrix}_{\gamma(t)}$$

so that for all $t \in [0, 2\pi r]$

$$\kappa(t) = \frac{1}{r}.$$

Therefore, a circle of radius r has signed curvature $1/r$ at all of its points. Notice that the circle

$$\delta : [0, 2\pi r] \to \mathbb{R}^2, \qquad t \mapsto (r\cos(t/r), -r\sin(t/r))$$

which travels clockwise around the origin $(0, 0) \in \mathbb{R}^2$ has signed curvature $-1/r$.

Remark 2.22

(i) When $\gamma : I \to \mathbb{R}^2$ is injective it is common to say that γ has curvature $\kappa(t)$ at the point $\gamma(t) \in \mathbb{R}^2$.

(ii) Whenever the acceleration vector is to the left of the velocity vector, the curve bends counter clockwise and the signed curvature is positive. Whenever the acceleration vector is to the right of the velocity vector, the curve bends clockwise and the signed curvature is negative.

It is desirable to also have a notion of curvature for a smooth immersed curve $\gamma : [a, b] \to \mathbb{R}^2$ which does not necessarily have unit speed. We can derive such a formula by computing the acceleration of the reparametrisation $\xi = \gamma \circ s^{-1} : [0, L] \to \mathbb{R}^2$ by arc-length $s : [a, b] \to [0, L]$ of γ. In the proof of Proposition 2.17 we obtained the formula

$$\dot{\xi}(u) = \frac{1}{\|\dot{\gamma}(s^{-1}(u))\|}\dot{\gamma}(s^{-1}(u))$$

[1] Don't confuse J_p with the Jacobian of a map!

which holds for all $u \in [0, L]$. Writing $t = s^{-1}(u)$, we equivalently have

$$\dot{\xi}(s(t)) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}$$

where $t \in [a, b]$. Computing the time derivative of the previous equation, we obtain

$$\begin{aligned}
\ddot{\xi}(s(t))s'(t) = \ddot{\xi}(s(t))\|\dot{\gamma}(t)\| \\
&= \frac{1}{\|\dot{\gamma}(t)\|}\ddot{\gamma}(t) + \dot{\gamma}(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\|\dot{\gamma}(t)\|}\right)(t) \\
&= \frac{1}{\|\dot{\gamma}(t)\|}\ddot{\gamma}(t) - \frac{\langle \ddot{\gamma}(t), \dot{\gamma}(t)\rangle}{\|\dot{\gamma}(t)\|^3}\dot{\gamma}(t)
\end{aligned}$$

so that in summary we have

$$\begin{aligned}
\dot{\xi}(u) &= \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|} \\
\ddot{\xi}(u) &= \frac{1}{\|\dot{\gamma}(t)\|^2}\left(\ddot{\gamma}(t) - \frac{\langle \ddot{\gamma}(t), \dot{\gamma}(t)\rangle}{\|\dot{\gamma}(t)\|^2}\dot{\gamma}(t)\right),
\end{aligned} \tag{2.7}$$

where again we write $u = s(t)$. Since $\|\dot{\xi}(u)\| = 1$ for all $u \in [0, L]$, we have

$$\frac{\langle \ddot{\xi}(u), J(\dot{\xi}(u))\rangle}{\|\dot{\xi}(u)\|^3} = \langle \ddot{\xi}(u), J(\dot{\xi}(u))\rangle = \frac{\langle \ddot{\gamma}(t), J(\dot{\gamma}(t))\rangle}{\|\dot{\gamma}(t)\|^3}, \tag{2.8}$$

where we use (2.7) and that $\langle \dot{\gamma}(t), J(\dot{\gamma}(t))\rangle = 0$.

Definition 2.23 (Curvature of a plane curve) Let $\gamma = (\gamma_1, \gamma_2) : I \to \mathbb{R}^2$ be a smooth curve. The function

$$\kappa : I \to \mathbb{R}, \qquad t \mapsto \frac{\langle \ddot{\gamma}(t), J(\dot{\gamma}(t))\rangle}{\|\dot{\gamma}(t)\|^3} = \frac{\gamma_1'(t)\gamma_2''(t) - \gamma_2'(t)\gamma_1''(t)}{(\gamma_1'(t)^2 + \gamma_2'(t)^2)^{3/2}} \tag{2.9}$$

is called the signed curvature of γ. The function

$$k : I \to [0, \infty), \qquad t \mapsto |\kappa(t)|$$

is called the curvature of γ. For all $t \in I$, the values $\kappa(t)$ and $k(t)$ are called the *signed curvature and curvature of γ at t*, respectively.

Remark 2.24 The motivation for the definition of the signed curvature as in (2.9) is (2.8). This equation states that if $\gamma : I \to \mathbb{R}^2$ is a smooth immersed curve with signed curvature $\kappa : I \to \mathbb{R}$, then the signed curvature $\hat{\kappa} : I \to \mathbb{R}^2$ of the parametrisation $\xi = \gamma \circ s^{-1}$ by arc length of γ satisfies

$$\hat{\kappa}(s(t)) = \kappa(t), \tag{2.10}$$

for all $t \in I$. Since $\xi(s(t)) = \gamma(t)$, we see that ξ and γ have the same signed curvature at $p = \gamma(t) = \xi(s(t))$. Observe that (2.5) implies that (2.9) agrees with the definition of curvature for a unit speed curve. We have thus found a notion of curvature for a plane curve which is unchanged – in the sense of (2.10) – after reparametrisation by arc length and which agrees with the definition of curvature for a unit speed curve. It thus is the natural definition of curvature for an immersed curve which does not necessarily have unit speed.

As a special case consider a smooth function $h : I \to \mathbb{R}$ and the associated smooth immersed curve

$$\gamma : I \to \mathbb{R}^2, \qquad t \mapsto (t, h(t))$$

whose image $\gamma(I)$ is the graph of h. In this case we have $\gamma_1(t) = t$ and $\gamma_2(t) = h(t)$ for all $t \in I$. Consequently, (2.9) gives

(2.11) $$\kappa(t) = \frac{h''(t)}{(1 + h'(t)^2)^{3/2}}.$$

Example 2.25 (Curvature of the graph of the sine function) The smooth immersed curve $\gamma : [0, 2\pi] \to \mathbb{R}^2$ associated to the graph of $\sin : [0, 2\pi] \to \mathbb{R}^2$ has signed curvature

$$\kappa(t) = -\frac{\sin(x)}{(1 + \cos(t)^2)^{3/2}}.$$

Example 2.26 (Curvature of the figure 8 curve) Let

$$\gamma : [0, 2\pi] \to \mathbb{R}^2, \qquad t \mapsto (\sin(t), \sin(t)\cos(t)).$$

The curve has velocity vectors

$$\dot{\gamma}(t) = \begin{pmatrix} \cos(t) \\ \cos(2t) \end{pmatrix}_{c(t)}$$

and acceleration vectors

$$\ddot{\gamma}(t) = \begin{pmatrix} -\sin(t) \\ -2\sin(2t) \end{pmatrix}_{c(t)}$$

for all $t \in [0, 2\pi]$. The signed curvature is thus given by

$$\kappa(t) = \frac{\cos(2t)\sin(t) - 2\cos(t)\sin(2t)}{(\cos(t)^2 + \cos(2t)^2)^{3/2}} = -\frac{3\sin(t) + \sin(3t)}{2\,(\cos(t)^2 + \cos(2t)^2)^{3/2}}$$

for all $t \in [0, 2\pi]$.

2.4 Local geometric properties of plane curves

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For a smooth immersed curve $\gamma = (\gamma_1, \gamma_2) : I \to \mathbb{R}^2$ we define

$$T : I \to T\mathbb{R}^2, \qquad t \mapsto T(t) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}$$

and

$$N : I \to T\mathbb{R}^2, \qquad t \mapsto N(t) = J(T(t)).$$

We call T the *unit tangent vector field along* γ and N *the unit normal vector field along* γ.

By construction, $\{T(t), N(t)\}$ forms an orthonormal basis of $T_{\gamma(t)}\mathbb{R}^2$ for all $t \in I$. A basis of some vector space is sometimes called a *frame* and the pair $\{T, N\}$ is called a *moving frame along* γ, since as time t progresses, the frame $\{T(t), N(t)\}$ moves along γ.

Suppose the signed curvature $\kappa : I \to \mathbb{R}$ of γ is non-vanishing for all $t \in I$ and define $\rho = 1/\kappa : I \to \mathbb{R}$. The curve

$$\delta : I \to \mathbb{R}^2, \qquad t \mapsto \delta(t) = E_{\gamma(t)}(\rho(t)N(t))$$

is called the *evolute of* γ. The circle with centre $\delta(t)$ and radius $r(t) = |\rho(t)|$ is called the *osculating circle of* γ *at* t. We will discuss the osculating circle more thoroughly in the exercises. Notice that

(2.12) $$\gamma(t) = E_{\delta(t)}(-\rho(t)N(t))$$

for all $t \in I$, where here we think of N as a vector field along δ.

In what follows we will assume that $\gamma = (\gamma_1, \gamma_2) : I \to \mathbb{R}^2$ has unit speed. In this case we obtain

$$T : I \to T\mathbb{R}^2, \qquad t \mapsto T(t) = \dot{\gamma}(t) = \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{pmatrix}_{\gamma(t)}$$

and

$$N : I \to T\mathbb{R}^2, \qquad t \mapsto N(t) = J(T(t)) = \begin{pmatrix} -\gamma_2'(t) \\ \gamma_1'(t) \end{pmatrix}_{\gamma(t)}.$$

We have the following equations known as the Frenet equations

$$\dot{T} = \kappa N, \qquad \text{and} \qquad \dot{N} = -\kappa T.$$

Written in "matrix notation" they become

$$\begin{pmatrix} \dot{T} \\ \dot{N} \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} T \\ N \end{pmatrix}. \tag{2.13}$$

Exercise 2.27 Derive the Frenet equations (2.13) for a unit speed curve $\gamma : I \to \mathbb{R}^2$.

Using the Frenet equations we can compute the velocity vector field of the evolute $\delta : I \to \mathbb{R}^2$ of a unit speed curve $\gamma : I \to \mathbb{R}^2$. Explicitly, we have for all $t \in \mathbb{R}$

$$\delta(t) = \left(\gamma_1(t) - \frac{\gamma_2'(t)}{\kappa(t)}, \gamma_2(t) + \frac{\gamma_1'(t)}{\kappa(t)}\right) = (\gamma_1(t) - \rho(t)\gamma_2'(t), \gamma_2(t) + \rho(t)\gamma_1'(t))$$

from which we compute

$$\dot{\delta}(t) = \begin{pmatrix} \gamma_1'(t) - \rho'(t)\gamma_2'(t) - \rho(t)\gamma_2''(t) \\ \gamma_2'(t) + \rho'(t)\gamma_1'(t) + \rho(t)\gamma_1''(t) \end{pmatrix}_{\delta(t)} = \rho'(t) \begin{pmatrix} -\gamma_2'(t) \\ \gamma_1'(t) \end{pmatrix}_{\delta(t)} = \rho'(t)N(t),$$

where we used the second Frenet equation $\dot{N}(t) = -\kappa(t)T(t)$, which is equivalent to

$$\begin{pmatrix} -\gamma_2''(t) \\ \gamma_1''(t) \end{pmatrix} = -\frac{1}{\rho(t)} \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{pmatrix}.$$

We can use the identity $\dot{\delta}(t) = \rho'(t)N(t)$ which holds for all $t \in I$ to show:

Theorem 2.28 (Plane curves of constant curvature) *Let $\gamma : I \to \mathbb{R}^2$ be a smooth immersed curve whose signed curvature $\kappa : I \to \mathbb{R}$ is constant, that is, there exists $c \in \mathbb{R}$ such that $\kappa(t) = c$ for all $t \in I$. Then either*

(i) $c \neq 0$ and $\gamma(I)$ is a segment of a circle of radius $1/c$;

(ii) $c = 0$ and $\gamma(I)$ is a segment of a line.

Proof Without loss of generality, by (2.8) we can assume that γ is a unit speed curve. Suppose $c \neq 0$. Since κ is constant, so is ρ and hence $\dot{\delta}(t) = 0$ for all $t \in I$. The velocity vector of δ thus vanishes for all $t \in I$ and therefore $\delta(I)$ consists of a single point $q \in \mathbb{R}^2$, that is, $\delta(t) = q$ for all $t \in I$. Since for all $t \in I$ the tangent vector $N(t)$ has length 1 and since $\rho(t) = 1/c$, (2.12) implies that all points of the curve γ have the same distance from q which means that $\gamma(I)$ is a segment of a circle of radius $1/c$.

Suppose $c = 0$. Then $\ddot{\gamma}(t) = 0$ for all $t \in I$ which is equivalent to

$$\gamma_1''(t) = \gamma_2''(t) = 0$$

for all $t \in I$. This implies that $\gamma(t) = (x_1 + tv_1, x_2 + tv_2) = E_p(t\vec{v}_p)$ for some point $p = (x_1, x_2) \in \mathbb{R}^2$ and tangent vector $\vec{v}_p = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_p$. Consequently, $\gamma(I)$ is a segment of a straight line. $\square$

It is natural to ask to what extent the signed curvature of a curve in $\mathbb{R}^2$ determines the curve. Phrased differently, can we recover the curve when we know its signed curvature?

Exercise 2.29 Let $\mathbf{R} \in \mathrm{O}(2)$ be an orthogonal 2×2-matrix, $q \in \mathbb{R}^2$ and $\kappa : I \to \mathbb{R}$ the signed curvature of a smooth immersed curve $\gamma : I \to \mathbb{R}^2$. Show that κ is invariant under Euclidean motion, that is, the curve

$$\delta : I \to \mathbb{R}^2, \qquad t \mapsto \delta(t) = f_{\mathbf{R},q}(\gamma(t))$$

has the same signed curvature as γ.

From Exercise 2.29 we learn that the curvature alone is not sufficient to determine the curve. We can however rule out Euclidean motions by specifying a point on the curve as well as T and N at this point. More precisely, we have:

Proposition 2.30 *Let $I = [a, b]$ be an interval. For a smooth function $\kappa : I \to \mathbb{R}$ there exists a unique smooth unit speed curve $\gamma : I \to \mathbb{R}^2$ such that $\gamma(a) = (0, 0) = 0_{\mathbb{R}^2}$ and*

$$T(a) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{0_{\mathbb{R}^2}} \quad \text{and} \quad N(a) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{0_{\mathbb{R}^2}} \tag{2.14}$$

and so that the signed curvature of γ is given by κ.

For the proof we need:

Lemma 2.31 *Let $\delta : [a, b] \to \mathbb{R}^2$ be a smooth curve with $\delta(t) \in S^1$ for all $t \in [a, b]$, where*

$$S^1 = \left\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\right\}.$$

Then there exists a smooth function $\phi : [a, b] \to \mathbb{R}$ – called a polar angle function *– so that for all $t \in [a, b]$ we have*

$$\delta(t) = (\cos(\phi(t)), \sin(\phi(t))).$$

Proof Let ϕ_0 be a real number so that $\delta(a) = (\cos(\phi_0), \sin(\phi_0))$. Clearly ϕ_0 is unique up to adding an integer multiple of 2π. We may define $\phi(t)$ as the sum of ϕ_0 and the distance travelled on S^1 from $\phi(a)$ to $\phi(t)$, where counter clockwise motion contributes positively and clockwise motion contributes negatively. First consider the case where δ moves counter clockwise – and counter clockwise only – around the unit circle S^1. In this case we can define

$$\phi(t) = \phi_0 + \int_a^t \|\dot{\delta}(w)\| \mathrm{d}w.$$

In general, δ may move clock wise as well and we can account for this as follows. Observe that there exists a unique smooth function $\xi : [a, b] \to \mathbb{R}$ so that

$$\begin{pmatrix} \delta_1'(t) \\ \delta_2'(t) \end{pmatrix} = \xi(t) \begin{pmatrix} -\delta_2(t) \\ \delta_1(t) \end{pmatrix}$$

for all $t \in [a, b]$. With this definition we have

$$\|\dot{\delta}(t)\| = |\xi(t)|\sqrt{\delta_1(t)^2 + \delta_2(t)^2} = |\xi(t)|.$$

Now define

$$\phi(t) = \phi_0 + \int_a^t \xi(w)\mathrm{d}w,$$

then $\phi : [a, b] \to \mathbb{R}$ is the desired *polar angle function*. □

Proof of Proposition 2.30 Let $\gamma = (\gamma_1, \gamma_2) : [a, b] \to \mathbb{R}^2$ be a unit speed curve. By the fundamental theorem of calculus we have

$$\gamma_1(t) = \int_a^t \gamma_1'(u)\mathrm{d}u + \mathrm{const}_1 \quad \text{and} \quad \gamma_2(t) = \int_a^t \gamma_2'(u)\mathrm{d}u + \mathrm{const}_2 \tag{2.15}$$

Recall that

$$T(t) = \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{pmatrix}_{\gamma(t)},$$

so that we can recover γ – up to translation in $\mathbb{R}^2$ by $(\mathrm{const}_1, \mathrm{const}_2)$ – from its unit tangent vector field.

By Lemma 2.31 there exists a polar angle function $\phi : [a, b] \to \mathbb{R}$ so that

$$T(t) = \begin{pmatrix} \cos(\phi(t)) \\ \sin(\phi(t)) \end{pmatrix}_{\gamma(t)} \quad \text{and} \quad N(t) = \begin{pmatrix} -\sin(\phi(t)) \\ \cos(\phi(t)) \end{pmatrix}_{\gamma(t)}$$

From this we compute using the Frenet equations

$$\dot{T}(t) = \begin{pmatrix} -\sin(\phi(t))\phi'(t) \\ \cos(\phi(t))\phi'(t) \end{pmatrix}_{\gamma(t)} = \kappa(t)\begin{pmatrix} -\sin(\phi(t)) \\ \cos(\phi(t)) \end{pmatrix}_{\gamma(t)},$$

so that $\phi'(t) = \kappa(t)$ for all $t \in [a, b]$. This gives the formula

$$\phi(t) = \int_a^t \kappa(w)\mathrm{d}w + \mathrm{const}. \tag{2.16}$$

Consequently, we can recover the unit tangent vector field – up to rotation by the angle const – from the signed curvature κ. Combining (2.15) and (2.16) we thus obtain the formulas

$$\gamma_1(t) = \int_a^t \cos\left(\int_a^u \kappa(w)\mathrm{d}w\right)\mathrm{d}u \quad \text{and} \quad \gamma_2(t) = \int_a^t \sin\left(\int_a^u \kappa(w)\mathrm{d}w\right)\mathrm{d}u.$$

These last two formulas uniquely determine γ up to the choice of integration constants. The conditions (2.14) precisely state that we have to choose all integration constants to be zero. □

2.5 Global geometric properties of plane curves

In order to compute the curvature of a smooth immersed curve $\gamma : [a, b] \to \mathbb{R}^2$ at time $t_0 \in [a, b]$, we only need to know the values of γ near t_0. We say that the curvature is a *local property* of a curve. Local properties are in contrast to *global properties* which try to capture geometric properties of the *whole curve*. The prototypical example of a global property of a plane curve is its total curvature:

Definition 2.32 (Total curvature of a plane curve) Let $\gamma : [a, b] \to \mathbb{R}^2$ be a smooth unit speed curve. The *total curvature of* γ is given by the integral of its signed curvature over the interval $[a, b]$.

$$\int_a^b \kappa(t)\mathrm{d}t.$$

A first observation about the total curvature is that it is *quantised*, that is, it is always an integer multiple of 2π, provided the curve is *closed*. Recall that a function $f : \mathbb{R} \to \mathbb{R}^m$ is called *periodic with period L* if $f(t+L) = f(t)$ for all $t \in \mathbb{R}$.

Definition 2.33 (Closed curve)

(i) Let $\gamma : [a, b] \to \mathbb{R}^m$ be a curve. Then γ is called *closed* if $\gamma(a) = \gamma(b)$.

(ii) Let $\gamma : [a, b] \to \mathbb{R}^m$ be a *smooth curve*. Then γ is called *closed* if there exists a smooth curve $\delta : \mathbb{R} \to \mathbb{R}^m$ which is periodic with period $(b - a)$ so that $\gamma(t) = \delta(t)$ for all $t \in [a, b]$.

Remark 2.34 Notice that if a smooth curve $\gamma : [a, b] \to \mathbb{R}^m$ is closed, then

$$\gamma^{(i)}(a) = \gamma^{(i)}(b).$$

for all $i \in \mathbb{N}$, that is, its derivatives agree to all orders at a and b.

Example 2.35 The “right half“ of the figure 8 curve

$$\gamma : [0, \pi] \to \mathbb{R}^2, \qquad t \mapsto (\cos(t), \sin(t)\cos(t))$$

is closed as a continuous curve, but not as a smooth curve, since

$$\gamma'(0) \neq \gamma'(\pi).$$

Recall that for the unit tangent vector field $T : [a, b] \to T\mathbb{R}^2$ of a smooth unit speed curve γ we have

$$T(t) = \begin{pmatrix} \cos(\phi(t)) \\ \sin(\phi(t)) \end{pmatrix}_{\gamma(t)}$$

where

$$\phi(t) = \int_a^t \kappa(w)\mathrm{d}w + \mathrm{const}$$

and $\kappa : [a, b] \to \mathbb{R}$ denotes the signed curvature of γ. If γ is closed, then $\gamma(a) = \gamma(b)$ and $T(a) = T(b)$ so that

$$T(a) = \begin{pmatrix} \cos(\phi(a)) \\ \sin(\phi(a)) \end{pmatrix}_{\gamma(a)} = T(b) = \begin{pmatrix} \cos(\phi(b)) \\ \sin(\phi(b)) \end{pmatrix}_{\gamma(b)}$$

This implies that $\phi(b) - \phi(a)$ is an integer multiple of 2π and hence

$$\frac{1}{2\pi}\int_a^b \kappa(t)\mathrm{d}t = \frac{1}{2\pi}(\phi(a) - \phi(b)) = N, \qquad N \in \mathbb{N}.$$

Definition 2.36 (Rotation index) Let $\gamma : [a, b] \to \mathbb{R}^2$ be a smooth closed unit speed curve with signed curvature $\kappa : [a, b] \to \mathbb{R}$. The integer

$$R_\gamma = \frac{1}{2\pi}\int_a^b \kappa(t)\mathrm{d}t$$

is called the *rotation index of* γ.

Observe that if $\gamma : [a,b] \to \mathbb{R}^2 \setminus \{(0,0)\}$ is a smooth curve, then the curve $\gamma : [a,b] \to \mathbb{R}^2$ defined by the rule

$$\delta(t) = \frac{1}{\sqrt{\gamma_1(t)^2 + \gamma_2(t)^2}} (\gamma_1(t), \gamma_2(t))$$

for all $t \in [a,b]$, is smooth and takes values in $S^1 \subset \mathbb{R}^2$. Hence by Lemma 2.31 we can write

$$\delta(t) = (\cos(\phi(t)), \sin(\phi(t)))$$

for some smooth polar angle function $\phi : [a,b] \to \mathbb{R}$. In the case where γ is closed, it follows as above that $(1/2\pi)(\phi(b) - \phi(a))$ is an integer known as the *winding number of* γ. It counts the total number of times that γ travels counter clockwise around the point $(0,0) \in \mathbb{R}^2$. A negative winding number indicates, that the curve travels clockwise around $(0,0)$.

Example 2.37 (Rotation index as winding number) The rotation index of a smooth closed unit speed curve $\gamma : [a,b] \to \mathbb{R}^2$ can be interpreted as the winding number of the first derivative $\gamma' : [a,b] \to \mathbb{R}^2 \setminus \{(0,0)\}$.

A closed curve which has no self intersections is called simple:

Definition 2.38 (Simple closed curve) A closed curve $\gamma : [a,b] \to \mathbb{R}^n$ is called *simple* if the restriction of γ to the half open interval $[a,b)$ is injective. Simple closed curves are often called *Jordan curves*.

Intuitively one might expect that the rotation index of a simple closed curve in the plane is either 1, in the case where the curve moves counter clockwise or -1, in the case where the curve moves clockwise. This is indeed true, but somewhat tricky to prove.

Theorem 2.39 *Let $\gamma : [0,L] \to \mathbb{R}^2$ be a smooth unit speed curve that is simple and closed. Then its rotation index is ± 1.*

This fact was probably already known to Riemann. We present a proof of H. Hopf.

Proof Without loss of generality we can assume that $\gamma(0) = (0,0)$ and that the image of γ is contained in $\{(x,y) \mid x \geqslant 0\}$. For $0 \leqslant s \leqslant t \leqslant L$ with $t - s < L$ denote by $\phi(s,t)$ the angle between $(\gamma(t) - \gamma(s))$ and $(1,0)$. Since γ is simple, $\gamma(t) - \gamma(s)$ is never equal to $(0,0)$. The function ϕ is uniquely determined by the condition to be continuous and that $|\phi(0,t)| \leqslant \pi/2$ for all $t \in (0,L)$. We also have $|\phi(s,L) - \pi| \leqslant \pi/2$ for all $s \in (0,L)$ and

$$\lim_{t \uparrow L} \phi(0,t) - \lim_{t \downarrow 0} \phi(0,t) = \lim_{s \uparrow L} \phi(s,L) - \lim_{s \downarrow 0} \phi(s,L) = \pm\pi.$$

Observe that the function

$$\phi(t) := \lim_{s \uparrow t} \phi(s,t) = \lim_{r \downarrow t} \phi(t,r)$$

is a continuous polar angle function for $\gamma' : [0,L] \to \mathbb{R}^2$, that is

$$\gamma'(t) = (\cos(\phi(t)), \sin(\phi(t)))$$

for all $t \in [0, L]$. Using $\phi(L) = \lim_{s\uparrow L} \phi(s, L)$ and $\phi(0) = \lim_{t\downarrow 0} \phi(0, t)$ as well as

$$\lim_{t\uparrow L} \phi(0, t) = \lim_{s\downarrow 0} \phi(s, L),$$

we compute

$$\begin{aligned}\int_0^L \kappa(t)\mathrm{d}t &= \phi(L) - \phi(0) = \lim_{s\uparrow L} \phi(s, L) - \lim_{t\downarrow 0} \phi(0, t)\\ &= \lim_{s\uparrow L} \phi(s, L) - \lim_{s\downarrow 0} \phi(s, L) + \lim_{s\downarrow 0} \phi(s, L) - \lim_{t\downarrow 0} \phi(0, t)\\ &= \lim_{s\uparrow L} \phi(s, L) - \lim_{s\downarrow 0} \phi(s, L) + \lim_{t\uparrow L} \phi(0, t) - \lim_{t\downarrow 0} \phi(0, t)\\ &= \pm\pi \pm \pi = \pm 2\pi,\end{aligned}$$

as claimed. □

2.6 Curves in three-dimensional space

The Frenet frame along a smooth unit speed curve in $\mathbb{R}^2$ assigns an orthonormal basis to every tangent space along γ. For a smooth unit speed curve $\gamma : I \to \mathbb{R}^3$ into three-dimensional space we can carry out a similar construction, provided the second derivative $\gamma'' : I \to \mathbb{R}^3$ is non-vanishing for all $t \in I$. For such a curve – called a *Frenet curve* – we define the *unit tangent vector field*

$$T : I \to T\mathbb{R}^3, \qquad t \mapsto T(t) := \dot\gamma(t)$$

the *unit normal vector field*

$$N : I \to T\mathbb{R}^3, \qquad t \mapsto N(t) = \frac{\dot T(t)}{\|\dot T(t)\|}$$

and the *unit binormal vector field*

$$B : I \to T\mathbb{R}^3, \qquad t \mapsto B(t) = T(t) \times N(t),$$

where we think of the cross product $\times$ as a map $T_{\gamma(t)}\mathbb{R}^3 \times T_{\gamma(t)}\mathbb{R}^3 \to T_{\gamma(t)}\mathbb{R}^3$ for all $t \in I$.

Definition 2.40 For a smooth immersed curve $\gamma = (\gamma_1, \gamma_2, \gamma_3) : I \to \mathbb{R}^3$ satisfying $\dot\gamma(t) \times \ddot\gamma(t) \neq 0_{T_{\gamma(t)}\mathbb{R}^3}$ for all $t \in I$, we define the *curvature* $\kappa : I \to \mathbb{R}$ and *torsion* $\tau : I \to \mathbb{R}$ by

$$\kappa(t) = \frac{\|\dot\gamma(t) \times \ddot\gamma(t)\|}{\|\dot\gamma(t)\|^3} \quad \text{and} \quad \tau(t) = \frac{\langle\dot\gamma(t), \ddot\gamma(t) \times \dddot\gamma(t)\rangle}{\|\dot\gamma(t) \times \ddot\gamma(t)\|^2},$$

Exercise 2.41

(i) Given a Frenet curve $\gamma : I \to \mathbb{R}^3$. Show that the Frenet equations

$$\begin{pmatrix}\dot T\\ \dot N\\ \dot B\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}T\\ N\\ B\end{pmatrix}.$$

hold.

(ii) Let $\mathbf{R} \in \mathrm{O}(3)$ and $q \in \mathbb{R}^3$. Show that the curvature and torsion are invariant under Euclidean motions. That is, if $\gamma : I \to \mathbb{R}^3$ is a smooth immersed curve with $\dot\gamma(t) \times \ddot\gamma(t) \neq 0_{T_{\gamma(t)}\mathbb{R}^3}$ and curvature $\kappa : I \to \mathbb{R}$ and torsion $\kappa : I \to \mathbb{R}$, then the curve

$$\delta = f_{\mathbf{R},q} \circ \gamma$$

has the same curvature and torsion as γ.

Similar to the case of plane curves we have:

Proposition 2.42 *Let $I = [a, b]$ be an interval. For smooth functions $\kappa : I \to \mathbb{R}^+$ and $\tau : I \to \mathbb{R}$ there exists a unique Frenet curve $\gamma : I \to \mathbb{R}^3$ with $\gamma(a) = (0, 0, 0) \in \mathbb{R}^3$ and*

$$T(a) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{0_{\mathbb{R}^3}} , \qquad N(a) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{0_{\mathbb{R}^3}} , \qquad B(a) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{0_{\mathbb{R}^3}}$$

and so that the curvature and torsion of γ are given by κ and τ, respectively.

In order to prove this fact one needs to solve a system of ordinary differential equations (Proposition 2.42 and its proof are not examinable).

Remark 2.43 There is also a notion of Frenet curve into $\mathbb{R}^m$ for $m > 3$. We refer to the literature for further details.

CHAPTER 3

Surfaces

3.1 Embedded surfaces

WEEK 4

In Linear Algebra I you saw the notion of the kernel of a linear map $f : V \to W$ between vector spaces V, W. A related notion is that of a level set. Here, for geometric concreteness, we restrict ourselves to level sets in $\mathbb{R}^3$, but the notion makes sense in any dimension.

Definition 3.1 Let $\mathcal{X} \subset \mathbb{R}^3$ be a set and $f : \mathcal{X} \to \mathbb{R}$ a function. The *level set of f with level* $c \in \mathbb{R}$ is the subset of $\mathcal{X}$ given by

$$f^{-1}(\{c\}) = \{p \in \mathcal{X} \mid f(p) = c\}.$$

Example 3.2 (2-plane) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a linear function.

(i) The kernel of f is the level set of f with level zero, that is, $\mathrm{Ker}(f) = f^{-1}(\{0\})$. If f has rank 1, then $f^{-1}(\{0\})$ has dimension 2 by the rank-nullity theorem and hence is a two-dimensional plane through the origin $0_{\mathbb{R}^3}$.

(ii) Let $c \neq 0$ be different from zero. Then $f^{-1}(\{c\})$ is an affine subspace whose associated vector space is $f^{-1}(\{0\})$

$$f^{-1}(\{c\}) = f^{-1}(\{0\}) + q = \{p + q \mid p \in \mathrm{Ker}(f)\},$$

where $q \in \mathbb{R}^3$ satisfies $f(q) = c$.

A 2-plane is not a particularly interesting object from the point of view of geometry. However, we obtain more interesting *surfaces* once we consider level sets arising from non-linear functions.

Example 3.3 (2-sphere) For $p = (x, y, z)$ we consider

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad p \mapsto x^2 + y^2 + z^2.$$

Then for all $r > 0$ the level set $f^{-1}(\{r^2\})$ of f with level r^2 is the 2-*sphere of radius r centred at* $0_{\mathbb{R}^3}$. We will denote it by $S^2(r)$ with the convention of writing S^2 when $r = 1$.

Example 3.4 (Torus) Let $R > 0$ and $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by the rule

$$p = (x, y, z) \mapsto \left(R - \sqrt{x^2 + y^2}\right)^2 + z^2.$$

Then for $r < R$ we consider the level set $f^{-1}(\{r^2\})$ of f with level r^2. This level set is called a *torus*.

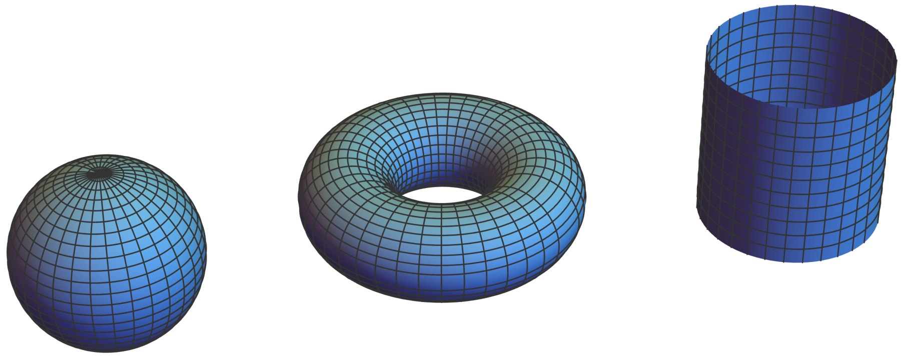

FIGURE 3.1. A sphere, a torus and a cylinder.

Example 3.5 (Cylinder) Consider the smooth function

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad p = (x, y, z) \mapsto x^2 + y^2$$

Then for $r > 0$ the level set $f^{-1}\left(\{r^2\}\right)$ of f with level r^2 is an (infinite) cylinder of radius r and central axis $\{(0, 0, z) \,|\, z \in \mathbb{R}\}$.

Example 3.6 (Paraboloid) For $a, b \in \mathbb{R}^+$ and $p = (x, y, z)$ consider $f : \mathbb{R}^3 \to \mathbb{R}$ defined by the rule

$$p \mapsto \frac{x^2}{a^2} + \frac{y^2}{b^2} - z.$$

The level set $f^{-1}(\{0\})$ of f with level zero is known as an elliptic paraboloid.

3.2 Tangent planes

For the 2-sphere $S^2(r) \subset \mathbb{R}^3$ we have an intuitive geometric understanding of what the tangent plane at $p = (x, y, z) \in S^2(r)$ is, namely the subspace of $T_p\mathbb{R}^3$ consisting of those vectors $\vec{v}_p$ where $\vec{v}$ is orthogonal to the line passing through the points p and $0_{\mathbb{R}^3}$. That is,

$$T_pS^2(r) = \left\{\vec{v}_p \in T_p\mathbb{R}^3 \,|\, v_1x + v_2y + v_3z = 0\right\} \subset T_p\mathbb{R}^3,$$

where

$$\vec{v}_p = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_p$$

It is natural to ask how we might define the tangent plane to a point $p \in f^{-1}(\{c\})$ for some level set defined by a function $f : \mathcal{X} \to \mathbb{R}$. The following example shows that f needs to satisfy certain conditions so that we obtain a geometrically natural definition of the tangent plane to a point.

Example 3.7 (Half-cone) For $c \in \mathbb{R}^+$ and $p = (x, y, z)$ consider $f : \mathbb{R}^3 \to \mathbb{R}$ defined by the rule

$$p \mapsto \frac{x^2}{c^2} + \frac{y^2}{c^2} - z^2.$$

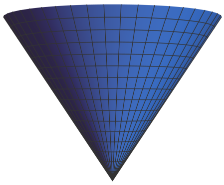

FIGURE 3.2. A half-cone. At the vertex of the cone we cannot define the tangent plane

Let $\mathcal{X} = \{p = (x, y, z) \in \mathbb{R}^3 \mid z \geqslant 0\}$ and consider the level set $C = f^{-1}(\{0\}) \cap \mathcal{X}$. Then C is a cone whose *vertex* (its tip) is $0_{\mathbb{R}^3}$. Clearly, we cannot define a tangent plane at the vertex of the cone in any geometrically natural way. Observe that f is smooth and that the exterior derivative of f is given by

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z = \frac{2x}{c^2}\mathrm{d}x + \frac{2y}{c^2}\mathrm{d}y - 2z\mathrm{d}z.$$

Recall that $\mathrm{d}f|_p : T_p\mathbb{R}^3 \to \mathbb{R}$ is a linear map satisfying

$$\mathrm{d}f|_p(\vec{v}_p) = \frac{2x}{c^2}v_1 + \frac{2y}{c^2}v_2 - 2zv_3,$$

where we write

$$\vec{v}_p = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_p$$

Therefore

$$\operatorname{rank} \mathrm{d}f|_p = \begin{cases} 1, & p \neq 0_{\mathbb{R}^3}, \\ 0, & p = 0_{\mathbb{R}^3}. \end{cases}$$

The rank of $\mathrm{d}f|_p$ fails to be maximal (i.e. 1) precisely at the vertex, where we cannot define the tangent plane.

This motivates the following definitions:

Definition 3.8 (Regular point and regular value) Let $f : U \to \mathbb{R}$ be a smooth function on the open set $U \subset \mathbb{R}^3$.

(i) A point $p \in U$ is called a *regular point of f* if $\mathrm{d}f|_p$ has rank 1.
(ii) A real number $c \in \mathbb{R}$ is called a *regular value or a regular level of f* if every point of $f^{-1}(\{c\})$ is a regular point of f.

Recall that we write $C^\infty(U, \mathbb{R})$ for the smooth functions on U.

Definition 3.9 (Smoothly embedded surface) Let $f \in C^\infty(U, \mathbb{R})$ and $c \in \mathbb{R}$ a regular value of f. Then we call

$$M = f^{-1}(\{c\}) \subset \mathbb{R}^3$$

a *smoothly embedded surface* in $\mathbb{R}^3$.

Remark 3.10

(i) We call the surface *embedded* since it is a subset of the larger *ambient space* $\mathbb{R}^3$.
(ii) As we will see later on, we can also consider a notion of a space which does not rely on an ambient space $\mathbb{R}^3$. Thus, there is a notion of *abstract space* – usually called a *manifold*.
(iii) We will often drop the adverb *smoothly* and simply speak of an *embedded surface* and hence implicitly we always assume that the surface arises as a level set of a smooth function.

Example 3.11 (Example 3.3 continued) For the 2-sphere we have

$$\mathrm{d}f = 2x\mathrm{d}x + 2y\mathrm{d}y + 2z\mathrm{d}z$$

so that $\mathrm{d}f|_p$ has rank 1 for all points $p \neq 0_{\mathbb{R}^3}$. Consequently, all $r \in \mathbb{R}^+$ are regular values of f. Since f is smooth we conclude that $S^2(r)$ is a smoothly embedded surface for all $r \in \mathbb{R}^+$. Observe that in this case we have

$$T_pS^2(r) = \mathrm{Ker}\left(\mathrm{d}f|_p\right). \tag{3.1}$$

for all $p \in S^2(r)$.

We use (3.1) as a motivation for the following definition:

Definition 3.12 (Tangent space and tangent bundle) Let $M = f^{-1}(\{c\})$ be an embedded surface.

(i) for all $p \in M$ the *tangent space of M at p* is defined by

$$T_pM = \mathrm{Ker}(\mathrm{d}f|_p) \subset T_p\mathbb{R}^3.$$

The elements of T_pM are said to be *tangent to M at p*.
(ii) By definition, for all $p \in M$ the tangent space T_pM is a subspace of $T_p\mathbb{R}^3$ whose dimension is

$$\dim T_pM = \dim T_p\mathbb{R}^3 - \dim \mathrm{Im}(\mathrm{d}f|_p) = 3 - \mathrm{rank}\,\mathrm{d}f|_p = 2,$$

by the rank–nullity theorem.
(iii) The *dimension* of M is the dimension of any tangent space of M, that is, 2.
(iv) The union of all tangent spaces is called the tangent bundle of M

$$TM = \bigcup_{p\in M} \left\{\vec{v}_p \in T_p\mathbb{R}^3 \,|\, \vec{v}_p \in \mathrm{Ker}(\mathrm{d}f|_p)\right\}.$$

Example 3.13 Write $p = (x, y, z)$ for a point in $\mathbb{R}^3$ and consider the linear function

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad p \mapsto z.$$

Then $M = f^{-1}(\{0\})$ is an embedded surface, the 2-dimensional vector subspace

$$M = \left\{p \in \mathbb{R}^3 \,|\, z = 0\right\} \subset \mathbb{R}^3$$

which is isomorphic to $\mathbb{R}^2$. The tangent space to $p \in M$ is

$$T_pM = \left\{\vec{v}_p \in T_p\mathbb{R}^3 \,|\, v_3 = 0\right\}.$$

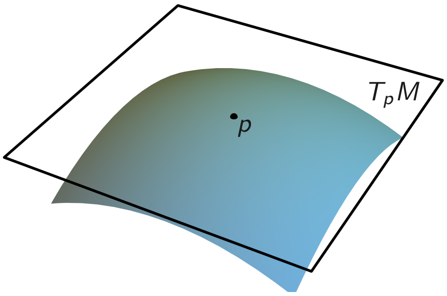

FIGURE 3.3. A piece of an embedded surface and its tangent plane at some point $p \in M$.

Simply forgetting about the third entry, we thus have a vector space isomorphism

$$T_pM \simeq T_{\hat{p}}\mathbb{R}^2$$

where $\hat{p}$ arises from p by deleting the third entry. The notion of the tangent space of $\mathbb{R}^2$ as defined in the first chapter is thus compatible with Definition 3.12 when we think of $\mathbb{R}^2$ as the embedded surface of $\mathbb{R}^3$ defined by $z = 0$.

Example 3.14 (Graph of a function) Let $U \subset \mathbb{R}^2$ be open and $h : U \to \mathbb{R}$ a smooth function. Then the graph of h

$$\mathcal{G}_h := \{(q, h(q)) \,|\, q \in U\}$$

is an embedded surface in $\mathbb{R}^3$. Indeed, consider $\mathcal{X} = U \times \mathbb{R} \subset \mathbb{R}^3$ and

$$f : \mathcal{X} \to \mathbb{R}, \qquad p = (q, t) \mapsto h(q) - t.$$

for $q \in U$ and $t \in \mathbb{R}$. Then

$$f^{-1}(\{0\}) = \{(q, h(q)) \,|\, q \in U\} = \mathcal{G}_h$$

and writing $q = (u, v)$, we have

$$\mathrm{d}f = \frac{\partial h}{\partial u}\mathrm{d}u + \frac{\partial h}{\partial v}\mathrm{d}v - \mathrm{d}t.$$

Therefore, $\mathrm{d}f|_{p=(q,t)}$ has rank 1 for all $p = (q, t) \in U \times \mathbb{R}$ and $M = f^{-1}(\{0\})$ is an embedded surface. The tangent space at $(q, h(q))$ for $q \in U$ is given by

$$T_{(q,h(q))}M = \left\{\vec{v}_{(q,h(q))} \in T_{(q,h(q))}\mathbb{R}^3 \;\middle|\; v_3 = \frac{\partial h}{\partial u}(q)v_1 + \frac{\partial h}{\partial v}(q)v_2\right\}.$$

Remark 3.15 (Gradient) Let $M = f^{-1}(\{c\})$ be an embedded surface. Recall that a subspace and its orthogonal complement are in direct sum. This implies that $\dim(T_pM)^\perp = 1$ for all $p \in M$ and since

$$\mathrm{d}f|_p(\vec{v}_p) = \langle \operatorname{grad} f(p), \vec{v}_p\rangle$$

we see that $\operatorname{grad} f(p)$ is a basis for $(T_pM)^\perp$ for all $p \in M$.

Definition 3.16 (Normal space and normal bundle) Let $M = f^{-1}(\{c\})$ be an embedded surface.

(i) For all $p \in M$, the orthogonal complement of $T_pM \subset T_p\mathbb{R}^3$ is called the *normal space to M at p*

$$T_pM^\perp = \mathrm{span}\,\{\mathrm{grad}\, f(p)\} \subset T_p\mathbb{R}^3.$$

(ii) The union of all normal spaces is called the normal bundle of M

$$TM^\perp = \bigcup_{p\in M} T_pM^\perp.$$

Remark 3.17 (Velocity vector of curves in a surface) Let $M = f^{-1}(\{c\}) \subset \mathbb{R}^3$ be an embedded surface. Suppose $\gamma : I \to \mathbb{R}^3$ is a smooth curve contained in M, that is, $\gamma(t) \in M$ for all $t \in I$. Then

$$\dot\gamma(t) \in T_{\gamma(t)}M.$$

for all $t \in I$. Indeed, since $\gamma(t) \in M$ for all $t \in M$, we have $f(\gamma(t)) = c$ for all $t \in I$. Taking the time t derivative of this equation we obtain

$$\mathrm{d}f|_{\gamma(t)}(\dot\gamma(t)) = 0$$

for all $t \in I$. This implies that $\dot\gamma(t)$ is tangent to M at $\gamma(t)$ for all $t \in I$.

3.3 Orientation

Definition 3.18 (Vector field and unit normal field)

(i) A *vector field on M* assigns to every point $p \in M$ an element of the tangent space T_pM at p, that is, it is a map

$$X : M \to TM \subset T\mathbb{R}^3$$

so that $X(p) \in T_pM$ for all $p \in M$.

(ii) A map

$$N : M \to TM^\perp \subset T\mathbb{R}^3$$

so that $N(p) \in T_pM^\perp$ and so that $\langle N(p), N(p)\rangle_p = 1$ for all $p \in M$ is called a *unit normal field* on M.

Writing a vector field or unit normal field on M as

$$a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$$

for functions $a, b, c : M \to \mathbb{R}$, the vector field or unit normal field is called *smooth* if the functions a, b, c are smooth in the sense of Remark 1.5.

Example 3.19 Let $M = f^{-1}(\{c\})$ be an embedded surface. The map

$$N : M \to TM^\perp, \qquad p \mapsto= \frac{\mathrm{grad}\, f(p)}{\|\mathrm{grad}\, f(p)\|}$$

is a smooth unit normal field on M.

Definition 3.20 (Orientation) Let $M = f^{-1}(\{c\})$ be an embedded surface. A choice of smooth unit normal vector field $N : M \to TM^{\perp}$ on M is called an *orientation*. An embedded surface equipped with a choice of orientation is called *oriented*.

3.4 Geodesics

Let $M = f^{-1}(\{c\})$ be an embedded surface.

Definition 3.21 (Geodesic) A smooth curve $\gamma : I \to M \subset \mathbb{R}^3$ is called a *geodesic in* M if

$$\ddot{\gamma}(t) \in T_{\gamma(t)}M^{\perp}$$

for all $t \in I$. That is, the acceleration vector $\ddot{\gamma}(t)$ is orthogonal to $T_{\gamma(t)}M$ for all $t \in I$.

Example 3.22 (Straight lines – Example 3.13 continued) Think of $\mathbb{R}^2$ as the embedded surface $M \subset \mathbb{R}^3$ consisting of those points $p = (x, y, z)$ for which $z = 0$. A curve γ in M is of the form

$$\gamma = (\gamma_1, \gamma_2, 0)$$

for smooth functions $\gamma_1, \gamma_2 : I \to \mathbb{R}$. Clearly $\ddot{\gamma}(t) \in T_{\gamma(t)}M^{\perp}$ if and only if $\ddot{\gamma}(t) = 0_{\mathbb{R}^3}$ for all $t \in I$. Therefore, the geodesics in $\mathbb{R}^2$ are segments of straight lines.

Example 3.23 (Helix on a cylinder) Consider the cylinder of radius r and central axis $\{(0, 0, z) \mid z \in \mathbb{R}\}$

$$M = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = r^2\right\} = f^{-1}\left(\{r^2\}\right),$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is given by $f(p) = x^2 + y^2$. For $b \in \mathbb{R}$ consider the helix

$$\gamma : \mathbb{R} \to M \subset \mathbb{R}^3, \qquad t \mapsto (r\cos(t), r\sin(t), bt).$$

Then, writing $p = (x, y, z)$ we have

$$\operatorname{grad} f(p) = \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix}_p$$

as well as

$$\dot{\gamma}(t) = \begin{pmatrix} -r\sin(t) \\ r\cos(t) \\ b \end{pmatrix}_{\gamma(t)} \quad \text{and} \quad \ddot{\gamma}(t) = -r\begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix}_{\gamma(t)} = -\frac{1}{2}\operatorname{grad} f(\gamma(t)).$$

Since $\operatorname{grad} f(p)$ is a basis of $T_pM^{\perp}$ for all $p \in M$, it follows that $\ddot{\gamma}(t) \in T_{\gamma(t)}M^{\perp}$ for all $t \in \mathbb{R}$, hence γ is a geodesic.

Example 3.24 (Great circle on a 2-sphere) The intersection of $S^2(r)$ with a 2-dimensional vector subspace $U \subset \mathbb{R}^3$ is called a *great circle*. Let $\{w_1, w_2\}$ be an orthonormal basis of U. Then $U \cap S^2(r)$ is the image of the curve

$$\gamma : \mathbb{R} \to S^2(r) \subset \mathbb{R}^3, \qquad t \mapsto r\cos(t)w_1 + r\sin(t)w_2.$$

Then

$$\ddot{\gamma}(t) = -r\left(\cos(t)\vec{w}_1 + \sin(t)\vec{w}_2\right)_{\gamma(t)}$$

where here $\vec{w}_i$ denotes the vector obtained by thinking of w_i as a column vector. Since $S^2(r) = f^{-1}\left(\{r^2\}\right)$ for the function $f : p = (x, y, z) \mapsto f(p) = x^2 + y^2 + z^2$, we have

$$\operatorname{grad} f(p) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}_p = 2\vec{p}_p$$

where again $\vec{p}$ denotes p, but thought of as a column vector. Consequently, we have

$$\operatorname{grad} f(\gamma(t)) = 2r\left(\cos(t)\vec{w}_1 + \sin(t)\vec{w}_2\right)_{\gamma(t)} = -2\ddot{\gamma}(t),$$

which shows that γ is a geodesic in $S^2(r)$.

Geodesics always have constant speed:

Proposition 3.25 *Let $\gamma : I \to M$ be a geodesic. Then $\|\dot{\gamma}(t)\|$ is independent of $t \in I$.*

Proof For a geodesic $\ddot{\gamma}(t)$ is always orthogonal to $\dot{\gamma}(t)$ and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\gamma}(t)\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle = 2\langle\dot{\gamma}(t), \ddot{\gamma}(t)\rangle = 0. \qquad \square$$

As we will see later on, geodesics are *locally length minimising* in the sense that if $\gamma : I \to M$ is a geodesic and $p, q \in \gamma(I)$ are points on the image of γ which are sufficiently close to each other, then the segment of the geodesic connecting p and q is the shortest curve in M which connects p and q.

Another interpretation of geodesics is in terms of the notion of a *free particle*. In classical mechanics, a *free particle* is a massive particle upon which no force acts. By Newton's second law of motion, a free particle has vanishing acceleration. A geodesic in an embedded surface M describes the movement of a particle that is not free in Newton's sense, but the force acting on it merely forces the particle to remain in M. The particle is *free in tangential directions*.

3.5 Covariant derivative

If $\gamma : I \to M$ is a smooth curve in an embedded surface, a map $X : I \to TM$ is called a vector field along γ if $X(t) \in T_{\gamma(t)}M$ for all $t \in M$, with smoothness defined as before. We would like to have a notion of derivative of a vector field along γ. If we take the usual time derivative of X, we obtain a map which in general takes values in $T\mathbb{R}^3$ and not TM. For instance, in the case of a smooth curve $\gamma : I \to M$, the velocity vector field $\dot{\gamma} : I \to TM$ is a vector field along γ, but its acceleration $\ddot{\gamma}$ is not, since $\ddot{\gamma}(t)$ does not necessarily lie in $T_{\gamma(t)}M$, but rather in $T_{\gamma(t)}\mathbb{R}^3$.

An obvious way to solve this problem is to compute the usual time derivative of a vector field along a curve and then apply an orthogonal projection onto each tangent space. More precisely:

Definition 3.26 (Covariant derivative) For a curve $\gamma : I \to M$ and a smooth vector field $X : I \to TM$ along γ, we define the *covariant derivative of* X as

$$\frac{\mathrm{D}X}{\mathrm{d}t}(t) := \Pi^{\perp}_{T_{\gamma(t)}M}(\dot{X}(t)),$$

where for $p \in M$

$$\Pi^{\perp}_{T_pM} : T_p\mathbb{R}^3 \to T_pM$$

denotes the orthogonal projection onto T_pM with respect to the inner product $\langle\cdot,\cdot\rangle_p$ on $T_p\mathbb{R}^3$ and where

$$\dot{X}(t) = \begin{pmatrix} X_1'(t) \\ X_2'(t) \\ X_3'(t) \end{pmatrix}_{\gamma(t)} \in T_{\gamma(t)}\mathbb{R}^3$$

with $X = \sum_{i=1}^3 X_i \frac{\partial}{\partial x_i}$ for smooth functions $X_i : I \to \mathbb{R}$.

Remark 3.27

(i) Notice that a smooth curve $\gamma : I \to M$ is a geodesic if and only if

$$\frac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t}(t) = 0$$

for all $t \in I$.

(ii) If $N : M \to TM^{\perp}$ is a smooth unit normal field on M and $X : I \to TM$ a smooth vector field along the curve $\gamma : I \to M$, then

$$\frac{\mathrm{D}X}{\mathrm{d}t}(t) = \dot{X}(t) - \langle N(\gamma(t)), \dot{X}(t)\rangle N(\gamma(t)).$$

Example 3.28 (Covariant derivative) Consider S^2 and γ to be the "equator"

$$\gamma : [0, 2\pi] \to S^2, \qquad t \mapsto (\cos(t), \sin(t), 0).$$

Observe that the vector fields along γ defined by the rule

$$E_1(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{pmatrix}_{\gamma(t)} \quad \text{and} \quad E_2(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{\gamma(t)}$$

span $T_{\gamma(t)}S^2$ for all $t \in [0, 2\pi]$. Furthermore

$$N(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix}_{\gamma(t)}$$

spans $(T_{\gamma(t)}S^2)^{\perp}$ for all $t \in [0, 2\pi]$ and $\{E_1(t), E_2(t), N(t)\}$ is an orthonormal basis of $T_{\gamma(t)}\mathbb{R}^3$ for all $t \in [0, 2\pi]$. Any smooth vector field $X : [0, 2\pi] \to TS^2$ along γ is of the form

$$X = s_1 E_1 + s_2 E_2$$

for smooth functions $s_1, s_2 : \mathbb{R} \to \mathbb{R}$ which are periodic with period 2π. From this we compute

$$\dot{X}(t) = s_1'(t)E_1(t) + s_2'(t)E_2(t) - s_1(t)N(t)$$

Hence we have

$$\frac{DX}{dt}(t) = s_1'(t)E_1(t) + s_2'(t)E_2(t) - s_1(t)N(t) - \langle s_1'(t)E_1(t) + s_2'(t)E_2(t) - s_1(t)N(t), N(t)\rangle N(t)$$

which simplifies to become

$$\frac{DX}{dt}(t) = s_1'(t)E_1(t) + s_2'(t)E_2(t).$$

Definition 3.29 (Parallel vector field along a curve) Let $\gamma : I \to M$ be a curve and $X : I \to TM$ a smooth vector field along γ. Then X is called *parallel along* γ if $\frac{DX}{dt}(t) = 0$ for all $t \in I$.

The velocity vector field of a geodesic γ is thus parallel along γ in the sense of the previous definition.

Exercise 3.30 Let $\gamma : [0, 2\pi] \to S^2$, $t \mapsto (\cos(t), \sin(t), 0)$ be the equator. Show that the vector fields E_1, E_2 along γ as defined above are parallel along γ.

Proposition 3.31 *Let $\gamma : I \to M$ be a curve and $X, Y : I \to TM$ smooth vector fields along γ and $u : I \to \mathbb{R}$ a smooth function. Then we have*

(i)

$$\frac{D}{dt}(X + Y)(t) = \frac{DX}{dt}(t) + \frac{DY}{dt}(t),$$

(ii)

$$\frac{D}{dt}(uX)(t) = u'(t)X(t) + u(t)\frac{DX}{dt}(t).$$

Proof This follows from the linearity of the usual derivative, the product rule for the derivative of real-valued functions and the definition of the covariant derivative. □

Remark 3.32 Observe that Proposition 3.31 and Exercise 3.30 immediately imply the end result of Example 3.28.

There are various questions related to geodesics. For instance, *how many* geodesics are there on an embedded surface? Do geodesics *keep moving* forever? We will come back to these questions later.

3.6 Curvature of embedded surfaces

WEEK 5

Given an embedded surface $M \subset \mathbb{R}^3$, we may ask how we can define a notion of curvature at each point of M.

Consider a plane $M = f^{-1}(\{0\}) \subset \mathbb{R}^3$, where $f : \mathbb{R}^3 \to \mathbb{R}$, $p = (x, y, z) \mapsto Ax + By + Cz$ is a linear function and the constants A, B, C are not all zero. In this case a unit normal field is given by

$$p \mapsto N(p) = \frac{1}{\sqrt{A^2+B^2+C^2}} \begin{pmatrix} A \\ B \\ C \end{pmatrix}_p .$$

Observe that this unit normal field is *constant* when we forget about the basepoint. That is, writing

$$N(p) = (\nu(p))_p$$

for some map $\nu : \mathbb{R}^3 \to M_{3,1}(\mathbb{R})$, we have

$$\nu(p) = \frac{1}{\sqrt{A^2+B^2+C^2}} \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

so that $\nu(p)$ is independent of p.

Intuitively a plane is a flat surface. In order to define a notion of curvature we can study how the unit normal field changes along an embedded surface. This leads to the notion of the shape operator.

Let $M = f^{-1}(\{c\})$ be an embedded surface and $N : M \to TM^{\perp}$ a unit normal field. We may take $N = \operatorname{grad} f / \|\operatorname{grad} f\|$. By assumption $\operatorname{grad} f(p)$ is non-zero for all points $p \in M$. Each point p of M admits an open neighbourhood on which $\operatorname{grad} f(p)$ is also non-zero. This implies that N is well defined on an open subset $U \subset \mathbb{R}^3$ which contains M. Again we write

$$N(p) = \nu(p)_p$$

for some function $\nu : U \to M_{3,1}(\mathbb{R})$. Explicitly we have

$$\nu(p) = \begin{pmatrix} \nu_1(p) \\ \nu_2(p) \\ \nu_3(p) \end{pmatrix}$$

for real-valued functions $\nu_i : U \to \mathbb{R}$. Since N is a unit normal field, we have

$$\sum_{i=1}^{3} \nu_i(p)^2 = 1$$

for all $p \in U$. Taking the exterior derivative of this equation, we obtain

$$0 = 2\sum_{i=1}^{3} \nu_i(p) \mathrm{d}\nu_i|_p. \tag{3.2}$$

Defining

$$[\mathrm{d}\nu(\vec{v}_p)]_p := \begin{pmatrix} \mathrm{d}\nu_1(\vec{v}_p) \\ \mathrm{d}\nu_2(\vec{v}_p) \\ \mathrm{d}\nu_3(\vec{v}_p) \end{pmatrix}_p$$

for all $\vec{v}_p \in TU$, (3.2) implies

$$\langle N(p), [\mathrm{d}\nu(\vec{v}_p)]_p \rangle = 0.$$

Recall that $T_pM^{\perp}$ is spanned by $N(p)$, hence $[\mathrm{d}\nu(\vec{v}_p)]_p$ is an element of T_pM for all $\vec{v}_p$. In particular, for all $p \in M$ we obtain a linear map

$$S_p : T_pM \to T_pM, \qquad \vec{v}_p \mapsto [\mathrm{d}\nu(\vec{v}_p)]_p.$$

Definition 3.33 (Shape operator and Gauss map) The map S_p is known as the *shape operator* and the map $\nu : M \to S^2 \subset M_{3,1}(\mathbb{R})$ is called the *Gauss map of M*.

Here S^2 denotes the 2-sphere in $M_{3,1}(\mathbb{R})$, that is, those column vectors $\vec{v} = (v_i)_{1 \leqslant i \leqslant 3}$ so that $(v_1)^2 + (v_2)^2 + (v_3)^2 = 1$.

Restricting $\langle \cdot, \cdot \rangle_p$ to $T_pM \subset T_p\mathbb{R}^3$, each tangent space of M is a Euclidean space and with respect to this Euclidean space structure we have the following important fact:

Proposition 3.34 *For all $p \in M$ the shape operator $S_p : T_pM \to T_pM$ is self-adjoint. That is, for all $p \in M$ and all $\vec{v}_p, \vec{w}_p \in T_pM$, we have*

$$\langle \vec{v}_p, S_p(\vec{w}_p) \rangle = \langle S_p(\vec{v}_p), \vec{w}_p \rangle.$$

We thus obtain two symmetric bilinear forms on each tangent space:

Definition 3.35 (First and second fundamental form) Let $M \subset \mathbb{R}^3$ be an embedded surface. The *first fundamental form of M at $p \in M$* is the restriction of the inner product on $T_p\mathbb{R}^3$ to T_pM, that is, we define

$$\mathrm{I}(\vec{v}_p, \vec{w}_p) := \langle \vec{v}_p, \vec{w}_p \rangle$$

for all $\vec{v}_p, \vec{w}_p \in T_pM$. The symmetric bilinear form on T_pM defined by the rule

$$\mathrm{II}(\vec{v}_p, \vec{w}_p) := -\langle S_p(\vec{v}_p), \vec{w}_p \rangle$$

for all $\vec{v}_p, \vec{w}_p \in T_pM$ is called the *second fundamental form of M at p*.

Proof of Proposition 3.34 Write $\xi = 1/\| \operatorname{grad} f \|$ and $p = (x_1, x_2, x_3)$. Then, for all $1 \leqslant i \leqslant 3$ we have $\nu_i = \xi \partial_i f$ and hence

$$\mathrm{d}\nu_i(\vec{v}_p) = \xi(p) \sum_{j=1}^{3} \partial_j \partial_i f(p) \mathrm{d}x_j(\vec{v}_p) + \partial_i f(p) \mathrm{d}\xi(\vec{v}_p).$$

Writing

$$\vec{v}_p = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_p \qquad \text{and} \qquad \vec{w}_p = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}_p,$$

we thus have

$$\langle S_p(\vec{v}_p), \vec{w}_p \rangle = \xi(p) \sum_{i=1}^{3} \sum_{j=1}^{3} (w_i \partial_j \partial_i f(p) \mathrm{d}x_j(\vec{v}_p)) + \mathrm{d}\xi(\vec{v}_p) \sum_{i=1}^{3} w_i \partial_i f(p).$$

Now notice that

$$\mathrm{d}\xi(\vec{v}_p) \sum_{i=1}^{3} w_i \partial_i f(p) = \mathrm{d}\xi(\vec{v}_p) \langle \operatorname{grad} f(p), \vec{w}_p \rangle = 0.$$

since $\vec{w}_p \in T_pM$ and $\operatorname{grad} f(p) \in T_pM^{\perp}$. Furthermore, $\mathrm{d}x_j(\vec{v}_p) = v_j$, hence we obtain

$$\langle S_p(\vec{v}_p), \vec{w}_p \rangle = \xi(p) \sum_{i=1}^{3} \sum_{j=1}^{3} w_i v_j \partial_j \partial_i f(p).$$

In terms of the *Hessian matrix* $\mathbf{H}_f(p)$ of f at p (whose entries are given by $[\mathbf{H}_f(p)]_{ij} = \partial_i \partial_j f(p)$), we can thus write

$$\mathrm{II}(\vec{v}_p, \vec{w}_p) = -\langle S_p(\vec{v}_p), \vec{w}_p \rangle = -\xi(p) \vec{w}^T \mathbf{H}_f(p) \vec{v}. \tag{3.3}$$

Since f is smooth the Hessian matrix is symmetric and this gives

$$\langle S_p(\vec{v}_p), \vec{w}_p \rangle = \xi(p) \vec{w}^T \mathbf{H}_f(p) \vec{v} = \langle \vec{v}_p, S_p(\vec{w}_p) \rangle,$$

as claimed. □

Example 3.36 (Shape operator of the 2-sphere) Let $M = S^2(r) \subset \mathbb{R}^3$ be the 2-sphere of radius $r > 0$. In this case $M = f^{-1}(\{r^2\})$ for the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $p = (x, y, z) \mapsto f(p) = x^2 + y^2 + z^2$. Clearly we have

$$\mathbf{H}_f(p) = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\xi(p) = \frac{1}{\|\operatorname{grad} f(p)\|} = \frac{1}{2r}.$$

Therefore, we have for all $p \in S^2(r)$ and $\vec{v}_p, \vec{w}_p \in T_pS^2$

$$\mathrm{II}(\vec{v}_p, \vec{w}_p) = -\langle S_p(\vec{v}_p), \vec{w}_p\rangle = -\frac{1}{r}\vec{v}^T\vec{w}.$$

It follows that

$$S_p(\vec{v}_p) = \frac{1}{r}\vec{v}_p.$$

Example 3.37 (Shape operator of a graph) Let I be an interval and $h : I \to \mathbb{R}$ a smooth function. We obtain an associated curve

$$\gamma : I \to \mathbb{R}^2, \qquad t \mapsto (t, h(t))$$

whose image is the graph $\mathcal{G}_h$ of h. We want to compute the shape operator of the graph of h. Recall that $\mathcal{G}_h$ is a level set with level 0 for the function

$$f : I \times \mathbb{R} \to \mathbb{R}, \qquad p = (t, s) \mapsto h(t) - s.$$

Clearly we have

$$\operatorname{grad} f(p) = \begin{pmatrix} h'(t) \\ -1 \end{pmatrix}_{(t,s)}$$

and hence

$$\mathbf{H}_f(p) = \begin{pmatrix} h''(t) & 0 \\ 0 & 0 \end{pmatrix}_{(t,s)}.$$

Let $p = (t, h(t))$ be an element of $M = \mathcal{G}_h$. A orthonormal basis of of T_pM is given by

$$\vec{e}_p = \frac{1}{\sqrt{1 + h'(t)^2}}\begin{pmatrix} 1 \\ h'(t) \end{pmatrix}_p$$

from which we compute

$$\langle S_p(\vec{e}_p), \vec{e}_p\rangle = \frac{h''(t)}{(1 + h'(t)^2)^{3/2}}. \tag{3.4}$$

Notice that T_pM is 1-dimensional, from this we conclude that

$$S_p(\vec{v}_p) = \frac{h''(t)}{(1 + h'(t)^2)^{3/2}}\vec{v}_p$$

for all $\vec{v}_p \in T_pM$, where $p = (t, h(t))$.

Definition 3.38 (Normal curvature) Let $M \subset \mathbb{R}^3$ be an embedded surface. Then for all $p \in M$ all all $\vec{v}_p \in T_pM$ with $\langle \vec{v}_p, \vec{v}_p\rangle = 1$, we define the *normal curvature at p in the direction $\vec{v}_p$* by

$$\kappa(\vec{v}_p) = -\mathrm{II}(\vec{v}_p, \vec{v}_p) = \langle S_p(\vec{v}_p), \vec{v}_p\rangle.$$

We would like to have a geometric interpretation of $\kappa(\vec{v}_p)$. By (3.4) the normal curvature of a graph $\mathcal{G}_h$ at $(t, h(t))$ is given by (both for $\vec{e}_p$ and for $-\vec{e}_p$)

$$\frac{h''(t)}{(1+h'(t)^2)^{3/2}}.$$

This is precisely the signed curvature at t of the curve $\gamma : I \to \mathbb{R}^2$, $t \mapsto (t, h(t))$.

It is tempting to try to interpret $\kappa(\vec{v}_p)$ as a signed curvature of a plane curve as well. To this end write

$$p = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \text{and} \qquad N(p) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_p \qquad \text{and} \qquad \vec{v}_p = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}_p$$

and consider the affine 2-plane $U_{\vec{v}_p} \subset \mathbb{R}^3$ passing through p and which is spanned by $N(p)$ and $\vec{v}_p$

$$U_{\vec{v}_p} := \{(x_1 + s_1 v_1 + s_2 w_1, x_2 + s_1 v_2 + s_2 w_2, x_3 + s_1 v_3 + s_2 w_3) \,|\, s_1, s_2 \in \mathbb{R}\}$$

The intersection of this affine 2-plane with M is a plane curve. Let $\epsilon > 0$ and $\gamma : (-\epsilon, \epsilon) \to U_{\vec{v}_p} \cap M$ be a smooth unit speed curve with $\gamma(0) = p$ and $\dot{\gamma}(0) = \vec{v}_p$ which is contained in the intersection $U_{\vec{v}_p} \cap M$. In order to apply the definition of the signed curvature of a plane curve, we choose a vector space isomorphism $\Psi : U_{\vec{v}_p} \to \mathbb{R}^2$ and compute the signed curvature of the curve $\delta := \Psi \circ \gamma : I \to \mathbb{R}^2$ at $t = 0$. Let $\Psi : U_{\vec{v}_p} \to \mathbb{R}^2$ be the vector space isomorphism so that

$$\Psi\left((x_1 + s_1 v_1 + s_2 w_1, x_2 + s_1 v_2 + s_2 w_2, x_3 + s_1 v_3 + s_2 w_3)\right) = (s_1, s_2)$$

for all $s_1, s_2 \in \mathbb{R}$. Observe that

$$\Psi_*(N(p)) = \vec{X} \qquad \text{and} \qquad \Psi_*(\vec{v}_p) = \vec{Y},$$

where here we simplify notation and write $\vec{X}, \vec{Y}$ for the standard basis of $T_{0_{\mathbb{R}^2}}\mathbb{R}^2$

$$\vec{X} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{0_{\mathbb{R}^2}}, \qquad \vec{Y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{0_{\mathbb{R}^2}}.$$

By definition, the signed curvature of δ at $t = 0$ is the real number κ_0 so that

$$\ddot{\delta}(0) = \kappa_0 J(\dot{\delta}(0)),$$

where $J : T_{0_{\mathbb{R}^2}}\mathbb{R}^2 \to T_{0_{\mathbb{R}^2}}\mathbb{R}^2$ is the unique linear map satisfying

$$J(\vec{X}) = \vec{Y} \qquad \text{and} \qquad J(\vec{Y}) = -\vec{X}.$$

Let $\langle\langle \cdot, \cdot \rangle\rangle$ denote the inner product on $T_{0_{\mathbb{R}^2}}\mathbb{R}^2$. Since $N(p), \vec{v}_p$ are orthonormal vectors in $T_p\mathbb{R}^3$ and $\vec{X}, \vec{Y}$ are orthonormal vectors in $T_{0_{\mathbb{R}^2}}\mathbb{R}^2$, it follows that

$$\Psi_* : \operatorname{span}\{\vec{v}_p, N(p)\} \to T_{0_{\mathbb{R}^2}}\mathbb{R}^2$$

is an orthogonal transformation. Using this fact we compute

$$\begin{aligned}
\langle \ddot{\gamma}(0), N(p) \rangle &= \langle\langle \Psi_*(\ddot{\gamma}(0)), \Psi_*(N(p)) \rangle\rangle = \langle\langle \ddot{\delta}(0), \vec{X} \rangle\rangle = -\kappa_0 \langle\langle J(\dot{\delta}(0), J(\vec{Y})) \\
&= -\kappa_0 \langle\langle \dot{\delta}(0), \vec{Y} \rangle\rangle = -\kappa_0 \langle\langle \Psi_*(\dot{\gamma}(0)), \vec{Y} \rangle\rangle \\
&= -\kappa_0 \langle\langle \Psi_*(\vec{v}_p), \vec{Y} \rangle\rangle = -\kappa_0 \langle\langle \vec{Y}, \vec{Y} \rangle\rangle = -\kappa_0.
\end{aligned}$$

We also have:

Lemma 3.39 *Let M be an embedded surface and $\gamma : (-\epsilon, \epsilon) \to M$ a smooth unit speed curve with $\gamma(0) = p$ and $\dot{\gamma}(0) = \vec{v}_p$, then*

$$\kappa(\vec{v}_p) = -\langle \ddot{\gamma}(0), N(p) \rangle.$$

Proof Let $\gamma = (\gamma_1, \gamma_2, \gamma_3) : I \to M \subset \mathbb{R}^3$ be a smooth unit speed curve with $\gamma(0) = p$ and $\dot{\gamma}(0) = \vec{v}_p$. Then

$$\begin{aligned}
0 &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \dot{\gamma}(t), N(\gamma(t)) \rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \sum_{i=1}^{3} \gamma_i'(t)\nu_i(\gamma(t)) \\
&= \sum_{i=1}^{3} \gamma_i''(0)\nu_i(\gamma(0)) + \sum_{i=1}^{3} \gamma_i'(0)\mathrm{d}\nu_i(\dot{\gamma}(0)) \\
&= \langle \ddot{\gamma}(0), N(p) \rangle + \langle S_p(\vec{v}_p), \vec{v}_p \rangle.
\end{aligned}$$

Hence we obtain the formula

$$\kappa(\vec{v}_p) = -\langle \ddot{\gamma}(0), N(p) \rangle.$$

□

In summary we see that

$$\kappa(\vec{v}_p) = -\langle \ddot{\gamma}(0), N(p) \rangle = \kappa_0$$

where κ_0 is the signed curvature at p of the curve cut out of M by the affine 2-plane $U_{\vec{v}_p}$ passing through p and which is spanned by $N(p)$ and $\vec{v}_p$. Notice that $\kappa(\vec{v}_p)$ depends on the choice of unit normal vector field N. Reversing the sign of $N(p)$ reverses the sign of $\kappa(\vec{v}_p)$.

Since $S_p : T_pM \to T_pM$ is self-adjoint for all $p \in M$, the *Spectral Theorem* from M06 Linear Algebra II implies that T_pM admits an ordered orthonormal basis $(\vec{v}_p, \vec{w}_p)$ consisting of eigenvectors of S_p.

Definition 3.40 (Principal curvatures and principal curvature directions) Let $M \subset \mathbb{R}^3$ be an embedded surface and $p \in M$. For $1 \leqslant i \leqslant 2$, the eigenvalues

$$\kappa_1(p) := \kappa(\vec{v}_p) = \langle S_p(\vec{v}_p), \vec{v}_p \rangle \qquad \text{and} \qquad \kappa_2(p) := \kappa(\vec{w}_p) = \langle S_p(\vec{w}_p), \vec{w}_p \rangle$$

of S_p at $p \in M$ are called the *principal curvatures of M at p.* The corresponding orthonormal eigenvectors $\vec{v}_p, \vec{w}_p$ are called the *principal curvature directions of M at p.*

The average trace of the shape operator is known as the mean curvature and the determinant of the shape operator as the Gauss curvature:

Definition 3.41 (Mean curvature and Gauss curvature) Let $M \subset \mathbb{R}^3$ be an embedded surface. We define

$$H : M \to \mathbb{R}, \qquad p \mapsto H(p) = \frac{1}{2}\operatorname{Tr} S_p = \frac{1}{2}\left(\kappa_1(p) + \kappa_2(p)\right).$$

We call $H(p)$ the *mean curvature of M at p*. We also define

$$K : M \to \mathbb{R}, \qquad p \mapsto K(p) = \det S_p = \kappa_1(p)\kappa_2(p).$$

We call $K(p)$ the *Gauss curvature of M at p*.

Using the principal curvatures we can classify the points of an embedded surface into different types:

Definition 3.42 Let $p \in M$ be a point of an embedded surface. Then p is called an *umbilical point* if $\kappa_1(p) = \kappa_2(p)$. If $\kappa_1(p) = \kappa_2(p) = 0$ we say p is a *planar point*. If

$K(p) = 0$, but $H(p) \neq 0$ we say p is a *parabolic point*. If $K(p) > 0$ we say p is an *elliptic point* and if $K(p) < 0$ we say p is a *hyperbolic point*.

Remark 3.43

(i) Notice that changing the sign of $N(p)$ changes the sign of $H(p)$, whereas the sign of $K(p)$ is unchanged.

(ii) Example 3.36 immediately implies that a 2-sphere of radius r has Gauss curvature $1/r^2$ and mean curvature $1/r$ at each of its points. It consists entirely of elliptic points.

(iii) An affine 2-plane in $\mathbb{R}^3$ has vanishing Gauss and mean curvature at each of its points. Unsurprisingly, it consists entirely of planar points.

(iv) With our convention of taking $N = \operatorname{grad} f/\|\operatorname{grad} f\|$, it follows that the normal curvature at $p \in M$ in the direction of $\vec{v}_p \in T_pM$ is *positive* if the surface *bends away* from N in the direction of $\vec{v}_p$ and it is *negative* if the surface *bends towards* N in the direction $\vec{v}_p$. In particular, $p \in M$ is an elliptical point if the surface *bends away* from N in both principal curvature directions or *bends towards* N in both principal curvature directions, whereas p is a hyperbolic point if it *bends towards* N in one principal curvature direction and *bends away* from N in the other.

Lemma 3.44 *Let $M \subset \mathbb{R}^3$ be an embedded surface, $p \in M$ and $\mathbf{b} = (\vec{v}_p, \vec{w}_p)$ an ordered orthonormal basis of T_pM. Then with respect to $\mathbf{b}$ the shape operator has matrix representation*

$$\mathbf{M}(S_p, \mathbf{b}, \mathbf{b}) = \begin{pmatrix} \langle S_p(\vec{v}_p), \vec{v}_p\rangle & \langle S_p(\vec{v}_p), \vec{w}_p\rangle \\ \langle S_p(\vec{w}_p), \vec{v}_p\rangle & \langle S_p(\vec{w}_p), \vec{w}_p\rangle \end{pmatrix}$$

Proof Exercise! □

Example 3.45 (Cylinder) Consider the cylinder of radius $r > 0$, which is the level set $M = f^{-1}(\{r^2\})$ with level r^2 of the function $f : \mathbb{R}^3 \to \mathbb{R}$, $p = (x, y, z) \mapsto x^2 + y^2$. Here we obtain

$$\operatorname{grad} f(p) = \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix}_p$$

and hence

$$\mathbf{H}_f(p) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

At $p = (x, y, z) \in M$ an ordered orthonormal basis of T_pM is given by $\mathbf{b} = (\vec{v}_p, \vec{w}_p)$, where

$$\vec{v}_p = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_p \qquad \text{and} \qquad \vec{w}_p = \frac{1}{r}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}_p$$

Using (3.3) and Lemma 3.44 we compute

$$\mathbf{M}(S_p, \mathbf{b}, \mathbf{b}) = \begin{pmatrix} 0 & 0 \\ 0 & 1/r \end{pmatrix}.$$

We conclude that $\vec{v}_p$ is a principal curvature direction with principal curvature 0, $\vec{w}_p$ is a principal curvature direction with principal curvature $1/r$. The Gauss curvature of the cylinder is $0 = 0 \cdot 1/r$ at each point of M and the mean curvature is $1/(2r)$ at each point of M. It follows that M consists entirely of parabolic points.

Example 3.46 (Hyperbolic paraboloid) Consider the hyperbolic paraboloid which is the level set with level 0 of the function $f : \mathbb{R}^3 \to \mathbb{R}$, $p = (x, y, z) \mapsto xy - z$. Here we obtain

$$\operatorname{grad} f(p) = \begin{pmatrix} y \\ x \\ -1 \end{pmatrix}_p$$

and hence

$$\mathbf{H}_f(p) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

At $p = (x, y, xy) \in M = f^{-1}(\{0\})$ an ordered orthonormal basis of T_pM is given by $\mathbf{b} = (\vec{v}_p, \vec{w}_p)$, where

$$\vec{v}_p = \frac{1}{\sqrt{1+y^2}} \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix}_p \qquad \text{and} \qquad \vec{w}_p = \frac{1}{\sqrt{1+x^2+y^2}} \begin{pmatrix} -xy/\sqrt{1+y^2} \\ \sqrt{1+y^2} \\ x/\sqrt{1+y^2} \end{pmatrix}_p .$$

Using (3.3) and Lemma 3.44 we compute

$$\mathbf{M}(S_p, \mathbf{b}, \mathbf{b}) = \frac{1}{1+x^2+y^2} \begin{pmatrix} 0 & 1 \\ 1 & -\frac{2xy}{\sqrt{1+x^2+y^2}} \end{pmatrix}$$

so that at $p = (x, y, xy) \in M$ we have Gauss curvature

$$K(p) = \det \mathbf{M}(S_p, \mathbf{b}, \mathbf{b}) = -\frac{1}{(1+x^2+y^2)^2} \tag{3.5}$$

and mean curvature

$$H(p) = \frac{1}{2} \operatorname{tr} \mathbf{M}(S_p, \mathbf{b}, \mathbf{b}) = -\frac{xy}{(1+x^2+y^2)^{3/2}}. \tag{3.6}$$

Notice that the Gauss curvature of M is negative at each point of M and hence M consists entirely of hyperbolic points.

Example 3.47 (Elliptic paraboloid) Consider an elliptic paraboloid which is the level set with level 0 of the function $f : \mathbb{R}^3 \to \mathbb{R}$, $p = (x, y, z) \mapsto \frac{x^2}{2} + \frac{y^2}{2} - z$. Here we obtain

$$\operatorname{grad} f(p) = \begin{pmatrix} x \\ y \\ -1 \end{pmatrix}_p$$

and

$$\mathbf{H}_f(p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

At $p = (x, y, (x^2+y^2)/2) \in M = f^{-1}(\{0\})$ an ordered orthonormal basis of T_pM is given by $\mathbf{b} = (\vec{v}_p, \vec{w}_p)$, where

$$\vec{v}_p = \frac{1}{\sqrt{1+x^2}} \begin{pmatrix} 1 \\ 0 \\ x \end{pmatrix}_p \qquad \text{and} \qquad \vec{w}_p = \frac{1}{\sqrt{1+x^2+y^2}} \begin{pmatrix} -xy/\sqrt{1+x^2} \\ \sqrt{1+x^2} \\ y/\sqrt{1+x^2} \end{pmatrix}_p .$$

Using (3.3) and Lemma 3.44 we compute

$$\mathbf{M}(S_p, \mathbf{b}, \mathbf{b}) = \frac{1}{(1+x^2)\sqrt{1+x^2+y^2}} \cdot \begin{pmatrix} 1 & -xy/\sqrt{1+x^2+y^2} \\ -xy/\sqrt{1+x^2+y^2} & (x^4+x^2(2+y^2)+1)/(1+x^2+y^2) \end{pmatrix}.$$

This gives

$$K(p) = \frac{1}{(1+x^2+y^2)^2} \qquad \text{and} \qquad H(p) = \frac{2+x^2+y^2}{2(1+x^2+y^2)^{3/2}}.$$

The Gauss curvature is positive at each point of M, hence M consists entirely of elliptic points.

Lemma 3.48 *Let $M \subset \mathbb{R}^3$ be an embedded surface, $p \in M$ and let $\kappa_1(p) \leqslant \kappa_2(p)$ denote the principal curvatures of M at p and $\vec{v}_p, \vec{w}_p$ the corresponding principal curvature directions. Then*

$$\kappa_1(p) = \min_{\vec{e}_p \in T_pM,\, \|\vec{e}_p\|=1} \kappa(\vec{e}_p) \qquad \kappa_2(p) = \max_{\vec{e}_p \in T_pM,\, \|\vec{e}_p\|=1} \kappa(\vec{e}_p)$$

Proof Every $\vec{e}_p \in T_pM$ with $\|\vec{e}_p\| = 1$ can be written as

$$\vec{e}_p = \cos(\alpha)\vec{v}_p + \sin(\alpha)\vec{w}_p$$

for some real number $\alpha \in \mathbb{R}$. From this we compute

$$\begin{aligned} \kappa(\vec{e}_p) &= \langle S_p(\cos(\alpha)\vec{v}_p + \sin(\alpha)\vec{w}_p), \cos(\alpha)\vec{v}_p + \sin(\alpha)\vec{w}_p\rangle \\ &= \cos(\alpha)^2\langle S_p(\vec{v}_p), \vec{v}_p\rangle + 2\cos(\alpha)\sin(\alpha)\langle S_p(\vec{v}_p), \vec{w}_p\rangle + \sin(\alpha)^2\langle S_p(\vec{w}_p), \vec{w}_p\rangle \\ &= \cos(\alpha)^2\kappa_1(p) + \sin(\alpha)^2\kappa_2(p). \end{aligned}$$

Since $\kappa_1(p) \leqslant \kappa_2(p)$ we obtain

$$\kappa_1(p) \leqslant \cos(\alpha)^2\kappa_1(p) + \sin(\alpha)^2\kappa_2(p) = \kappa(\vec{e}_p) \leqslant \kappa_2(p)$$

and $\kappa_1(p) = \kappa(\vec{e}_p)$ for the choice $\alpha = 0$ and $\kappa_2(p) = \kappa(\vec{e}_p)$ for the choice $\alpha = \pi/2$. □

Remark 3.49 Lemma 3.48 and the self-adjointness of the shape operator (Proposition 3.34) have quite a remarkable geometric consequence. Together they imply that when we pick any point p on an embedded surface M and determine the tangential directions $\vec{e}_p$ to M at p in which the surface bends the most and the least, then the two directions are always orthogonal.

3.7 Local parametrisations

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An affine 2-plane in $\mathbb{R}^3$ is the set M of solutions to an equation of the form

$$Ax + By + Cz = D$$

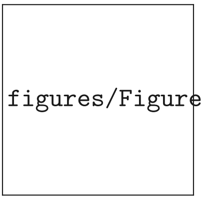

FIGURE 3.4. The image of the injective immersion $F : (0, 2\pi) \times (0, 1) \to \mathbb{R}^3$ defined by $(u, v) \mapsto (\sin(u), \sin(u)\cos(u), v)$.

for some coefficients A, B, C not all zero. We thus obtain the plane as the *level set* of the smooth function the defined by the rule $p = (x, y, z) \mapsto f(p) = Ax + By + Cz$, that is, $M = f^{-1}(\{D\})$.

Alternatively, we can describe the affine plane as the *image of a smooth map*

$$F : \mathbb{R}^2 \to \mathbb{R}^3, \qquad q = (u, v) \mapsto p_0 + uw_1 + vw_2,$$

for points $p_0, w_1, w_2 \in \mathbb{R}^3$ such that $f(p_0) = D$ and $f(w_1) = f(w_2) = 0$.

Many surfaces are much easier to describe as the image of a smooth map, rather than as a level set. Unfortunately, it is exceptional that the *whole* surface is the image of single map, as it is the case for an affine 2-plane in $\mathbb{R}^3$. In general one needs several maps that *parametrise a surface*. This leads to the notion of a local parametrisation.

Definition 3.50 (Local parametrisation of a surface) Given an embedded surface $M \subset \mathbb{R}^3$, a smooth map $F : U \to \mathbb{R}^3$ defined on some open subset $U \subset \mathbb{R}^2$ so that

(i) $\mathrm{Im}(F) \subset M$;
(ii) F is injective;
(iii) F is an *immersion*. This means that $F_*|_q : T_qU \to T_{F(q)}\mathbb{R}^3$ is injective for all $q \in U$;
(iv) F is a homeomorphism onto its image. This means that there exists an open subset W of $\mathbb{R}^3$ which contains the image of F and a continuous map $\Phi : W \to U$ so that $\Phi(F(q)) = q$ for all $q \in U$;

is called a *local parametrisation of M* (or more precisely a local parametrisation of $\mathrm{Im}(F)$). The restriction of Φ to $W \cap M$ is called a *local coordinate system on $W \cap M$*.

Exercise 3.51 Consider the injective immersion $F : (0, 2\pi) \times (0, 1) \to \mathbb{R}^3$ defined by $(u, v) \mapsto (\sin(u), \sin(u)\cos(u), v)$. Show that F is not a homeomorphism onto its image (compare Figure 4.1).

Exercise 3.52 Write $q = (u, v)$ for a point of $U \subset \mathbb{R}^2$. Show that $F_*|_q$ is injective if and only if the cross product

$$\frac{\partial F}{\partial u}(q) \times \frac{\partial F}{\partial v}(q) := \begin{pmatrix} \frac{\partial F_1}{\partial u}(q) \\ \frac{\partial F_2}{\partial u}(q) \\ \frac{\partial F_3}{\partial u}(q) \end{pmatrix} \times \begin{pmatrix} \frac{\partial F_1}{\partial v}(q) \\ \frac{\partial F_2}{\partial v}(q) \\ \frac{\partial F_3}{\partial v}(q) \end{pmatrix}$$

is non-vanishing, where we write $F = (F_1, F_2, F_3)$ for smooth functions $F_i : U \to \mathbb{R}$.

Example 3.53 (Stereographic projection) Consider the 2-sphere S^2 and the map

$$F : \mathbb{R}^2 \to \mathbb{R}^3, \qquad q = (u, v) \mapsto \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2} \right)$$

Clearly F is smooth and injective and we have

$$\frac{\partial F}{\partial u}(q) = \frac{1}{(1+u^2+v^2)^2} \begin{pmatrix} 2-2u^2+2v^2 \\ -4uv \\ 4u \end{pmatrix}$$

and

$$\frac{\partial F}{\partial v}(q) = \frac{1}{(1+u^2+v^2)^2} \begin{pmatrix} -4uv \\ 2(1+u^2-v^2) \\ 4v \end{pmatrix}$$

From which we obtain

$$\frac{\partial F}{\partial u}(q) \times \frac{\partial F}{\partial v}(q) = -\frac{1}{(1+u^2+v^2)^3} \begin{pmatrix} 8u \\ 8v \\ 4(-1+u^2+v^2) \end{pmatrix} \neq 0_{\mathbb{R}^3}$$

so that $F : \mathbb{R}^2 \to \mathbb{R}^3$ is an immersion. Notice that the image of F is contained in $S^2 \subset \mathbb{R}^3$, it does however not contain the *north pole* $(0, 0, 1) \in S^2$. Consider the open subset of $\mathbb{R}^3$ defined by $W = \mathbb{R}^3 \setminus \{(x, y, 1) \,|\, x, y \in \mathbb{R}\}$ and

$$\Phi : W \to \mathbb{R}^2, \qquad p = (x, y, z) \mapsto \Phi(p) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right). \tag{3.7}$$

The map Φ is called the *stereographic projection from the north pole*. It is continuous and moreover, a direct calculation shows that $\Phi(F(q)) = q$ for all $q \in \mathbb{R}^2$. It follows that $F : \mathbb{R}^2 \to \mathbb{R}^3$ is a local parametrisation of S^2 with the north pole removed. Likewise, consider the map

$$\hat{F} : \mathbb{R}^2 \to \mathbb{R}^3, \qquad q = (u, v) \mapsto \left(\frac{2u}{1+u^2+v^2}, \frac{-2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2} \right)$$

As above, one chan check that $\hat{F}$ is a smooth injective immersion and defining $\hat{W} = \mathbb{R}^3 \setminus \{(x, y, -1) \,|\, x, y \in \mathbb{R}\}$ and

$$\hat{\Phi} : \hat{W} \to \mathbb{R}^2, \qquad p = (x, y, z) \mapsto \hat{\Phi}(p) = \left(\frac{x}{1+z}, \frac{-y}{1+z} \right).$$

we have $\hat{\Phi}(\hat{F}(p)) = p$ for all $p \in \mathbb{R}^2$. The map $\hat{\Phi}$ is called the *stereographic projection from the south pole* and the $\hat{F}$ is a local parametrisation of S^2 with the south pole removed. We conclude that we can parametrise S^2 in terms of two maps F and $\hat{F}$.

Exercise 3.54 Show that for a point $p = (x, y, z) \in S^2 \setminus \{(0, 0, 1)\}$, the *equatorial plane* $\{(x, y, 0) \,|\, x, y, \in \mathbb{R}\} \subset \mathbb{R}^3$ intersects the straight line through $(0, 0, 1)$ and p in the point

$$\left(\frac{x}{1-z}, \frac{y}{1-z}, 0 \right).$$

Because of this fact the map Ψ from (3.7) is called the *stereographic projection from the north pole*. Likewise, the straight line through $(0, 0, -1)$ and $p \in S^2 \setminus \{(0, 0, -1)\}$ intersects the equatorial plane in the point

$$\left(\frac{x}{1+z}, \frac{-y}{1+z}, 0 \right).$$

Another local parametrisation of the sphere is given by the following map:

Example 3.55 (Spherical coordinates) The 2-sphere $S^2(r)$ with half a *meridian* removed is parametrised by the map $F : (0, 2\pi) \times (-\pi/2, \pi/2) \to S^2(r) \subset \mathbb{R}^3$ defined by the rule

$$(u, v) \mapsto (r\cos(v)\cos(u), r\cos(v)\sin(u), r\sin(v)).$$

The coordinates associated with this parametrisation are known as *spherical coordinates*.

Example 3.56 (Torus) Recall that the torus is the level set of the function $f(p) = \left(R - \sqrt{x^2+y^2}\right)^2 + z^2$ with level r^2. A local parametrisation of the torus is given by the map $F : (0, 2\pi) \times (0, 2\pi) \to \mathbb{R}^3$ defined by the rule

$$(u, v) \mapsto ((R + r\cos(v))\cos(u), (R + r\cos(v))\sin(u), r\sin(v)).$$

Example 3.57 (Graph of a function) Let $U \subset \mathbb{R}^2$ be an open set and $h : U \to \mathbb{R}$ a smooth function. Recall that the graph $\mathcal{G}_h$ of h is an embedded surface. Consider

$$F : U \to \mathbb{R}^3, \qquad q = (u, v) \mapsto (u, v, h(u, v)).$$

Then F is smooth, injective and moreover an immersion since

$$\frac{\partial F}{\partial u}(q) \times \frac{\partial F}{\partial v}(q) = \begin{pmatrix} -\frac{\partial h}{\partial u}(q) \\ -\frac{\partial h}{\partial v}(q) \\ 1 \end{pmatrix} \neq 0_{\mathbb{R}^3}$$

Let $W = U \times \mathbb{R} \subset \mathbb{R}^3$ and define

$$\Phi : W \to \mathbb{R}^2, \qquad p = (x, y, z) \mapsto \Phi(p) = (x, y).$$

Clearly Φ is continuous and $\Phi(F(q)) = q$ for all $q \in U$. It follows that the graph $\mathcal{G}_h$ of h is parametrised by F.

Remark 3.58 Choosing the function $h(u, v) = uv$ in the previous example gives the hyperbolic paraboloid.

Having the notion of a local parametrisation of a surface, we should make sure that sufficiently small pieces of an embedded surface admit a local parametrisation. This is a consequence of the implicit function theorem.

Theorem 3.59 (Special case of the implicit function theorem) *Every embedded surface $M \subset \mathbb{R}^3$ is locally the composition of a Euclidean motion and the graph of a smooth function. That is, for every point $p \in M$ there exists a Euclidean motion $f_{\mathbf{R},q} : \mathbb{R}^3 \to \mathbb{R}^3$, an open set $W \subset \mathbb{R}^3$ containing $\mathbf{R}p + q$, an open set $U \subset \mathbb{R}^2$ and a smooth function $h : U \to \mathbb{R}$ so that $W \cap f_{\mathbf{R},q}(M) = \mathcal{G}_h$.*

Proof Notice that if $\mathbf{A}$ is an invertible 3×3-matrix and $b \in \mathbb{R}^3$, then $f_{\mathbf{A},b}(M)$ is also an embedded surface. Fix $p \in M$ and choose a Euclidean motion $f_{\mathbf{R},q} : \mathbb{R}^3 \to \mathbb{R}^3$ so that

$f_{\mathbf{R},q}(p) = 0_{\mathbb{R}^3}$ and so that the tangent space of $\tilde{M} = f_{\mathbf{R},q}(M)$ at $0_{\mathbb{R}^3}$ is spanned by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{0_{\mathbb{R}^3}} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{0_{\mathbb{R}^3}} .$$

We want to argue that $\tilde{M}$ is locally a graph near $0_{\mathbb{R}^3}$. For $q = (u, v) \in \mathbb{R}^2$ consider the curve γ_q through $(u, v, 0)$ which is perpendicular to the plane $\{(x, y, 0) \,|x, y \in \mathbb{R}\}$

$$\gamma_q : \mathbb{R}^2 \to \mathbb{R}^3, \qquad q = (u, v) \mapsto \gamma_q(t) = (u, v, t)$$

Since the tangent plane of $\tilde{M}$ at $0_{\mathbb{R}^3}$ is horizontal, the curve γ_q will intersect $\tilde{M}$ for sufficiently small values of $(u, v) = q$. Mapping $q = (u, v)$ to the smallest (in absolute value) time t for which γ_q intersects $\tilde{M}$, we obtain a smooth real-valued function h on some open neighbourhood U of $0_{\mathbb{R}^2}$. By construction, $\tilde{M}$ is locally the graph of h, that is, there exists an open set $W \subset \mathbb{R}^3$ so that $W \cap \tilde{M} = \mathcal{G}_h$. □

Exercise 3.60 Show that Theorem 3.59 is still true when the Euclidean motion is replaced with $f_{\mathbf{P}_\sigma} : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbf{P}_\sigma \in M_{3,3}(\mathbb{R})$ denotes the permutation matrix of a permutation $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$.

Given an embedded surface $M \subset \mathbb{R}^3$, we can conclude from Theorem 3.59 that for each point $p \in M$ we can find an open set $W \subset \mathbb{R}^3$ containing p so that $W \cap M$ is parametrised by the map

$$F : U \to \mathbb{R}^3, \qquad q = (u, v) \mapsto f_{\mathbf{R},q}(u, v, h(u, v)),$$

where $f_{\mathbf{R},q} : \mathbb{R}^3 \to \mathbb{R}^3$ is a Euclidean motion and $h : U \to \mathbb{R}$ a smooth function defined on some open set $U \subset \mathbb{R}^2$. A sufficiently small piece of an embedded surface thus always admits a local parametrisation.

3.8 Calculations in local parametrisations

If $M = f^{-1}(\{c\})$ is an embedded surface and $F : U \to \mathbb{R}^3$ a local parametrisation of M, we have that $f(F(q)) = c$ for all $q \in U$. Taking the exterior derivative of this identity, we conclude that

$$\mathrm{d}f|_{F(q)}(F_*|_q(\vec{w}_q)) = 0$$

for all $\vec{w}_q \in T_qU$. Since $F_*|_q$ is injective, this means that

$$T_{F(q)}M = \{F_*|_q(\vec{w}_q) \,|\, \vec{w}_q \in T_qU\} .$$

That is, the linear map $F_*|_q : T_qU \to T_{F(q)}\mathbb{R}^3$ maps the tangent space of U at $q \in U$ onto the tangent space of M at $F(q)$. In particular, writing

$$\partial_1 F(q) := \frac{\partial F}{\partial u}(q) \qquad \text{and} \qquad \partial_2 F(q) := \frac{\partial F}{\partial v}(q)$$

it follows that

$$\mathbf{b}_{F(q)} = \left((\partial_1 F(q))_{F(q)}, (\partial_2 F(q))_{F(q)}\right)$$

is an ordered basis $\mathbf{b}_{F(q)}$ of $T_{F(q)}M$ for all $q \in U$.

With respect to a choice of local parametrisation F of M we can thus encode the first fundamental form I of M in terms of a map g on U with values in the symmetric 2×2-matrices $g : U \to M_{2,2}(\mathbb{R})$. The map g assigns to a point $q \in U$ the matrix representation of the inner product $\mathrm{I}_{F(q)} = \langle\cdot,\cdot\rangle_{F(q)}$ with respect to the ordered basis $\mathbf{b}_{F(q)}$

$$q \mapsto g(q) = \mathbf{M}(\mathrm{I}_{F(q)}, \mathbf{b}_{F(q)}) = \begin{pmatrix} \partial_1 F(q)\cdot\partial_1 F(q) & \partial_1 F(q)\cdot\partial_2 F(q) \\ \partial_2 F(q)\cdot\partial_1 F(q) & \partial_2 F(q)\cdot\partial_2 F(q) \end{pmatrix},$$

where $\cdot$ denotes the standard scalar product of column vectors. For $1 \leqslant i,j \leqslant 2$ we write $g_{ij}(q) = \left[\mathbf{M}(\langle\cdot,\cdot\rangle_{F(q)}, \mathbf{b}_{F(q)}\right]_{ij}$ so that

$$\boxed{\begin{aligned} g_{11}(q) &= \partial_1 F(q)\cdot\partial_1 F(q), \\ g_{12}(q) &= \partial_1 F(q)\cdot\partial_2 F(q) = \partial_2 F(q)\cdot\partial_1 F(q) = g_{21}(q), \\ g_{22}(q) &= \partial_2 F(q)\cdot\partial_2 F(q). \end{aligned}} \tag{3.8}$$

or written more succintly (while surpressing the base point)

$$g_{ij} = \partial_i F\cdot\partial_j F \tag{3.9}$$

Example 3.61 Consider the local parametrisation of the 2-sphere of radius 1 given in Example 3.55

$$F(u,v) = (\cos(v)\cos(u), \cos(v)\sin(u), \sin(v)).$$

In this case we obtain

$$\partial_1 F(q) = \begin{pmatrix} -\cos(v)\sin(u) \\ \cos(v)\cos(u) \\ 0 \end{pmatrix} \quad \text{and} \quad \partial_2 F(q) = \begin{pmatrix} -\sin(v)\cos(u) \\ -\sin(v)\sin(u) \\ \cos(v) \end{pmatrix}$$

where we write $q = (u,v)$. From this we compute

$$g_{11}(q) = (-\cos(v)\sin(u))^2 + (\cos(v)\cos(u))^2 + (0)^2 = \cos(v)^2$$

and likewise $g_{12} = 0$ and $g_{22} = 1$ so that

$$g(q) = \begin{pmatrix} \cos(v)^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Example 3.62 For the parametrisation of the hyperbolic paraboloid

$$F : \mathbb{R}^2 \to \mathbb{R}^3, \qquad q = (u,v) \mapsto (u, v, uv)$$

we obtain

$$g(q) = \begin{pmatrix} 1+v^2 & uv \\ uv & 1+u^2 \end{pmatrix}.$$

Exercise 3.63 Show that for the parametrisation of the torus given in Example 3.56 we obtain

$$g(q) = \begin{pmatrix} (R + r\cos(v))^2 & 0 \\ 0 & r^2 \end{pmatrix}$$

where we write $q = (u,v)$.

We can also encode the second fundamental form II of M in terms of a matrix-valued map on U. We define

$$A : U \to M_{2,2}(\mathbb{R}), \qquad q \mapsto A(q) = \mathbf{M}(\mathrm{II}_{F(q)}, \mathbf{b}_{F(q)}).$$

To compute the matrix entries of A explicitly, first observe that we may choose the unit normal field $N : M \to TM^{\perp}$ so that

$$G(q) := \nu(F(q)) = \frac{\partial_1 F(q) \times \partial_2 F(q)}{|\partial_1 F(q) \times \partial_2 F(q)|}, \tag{3.10}$$

where we write $N(p) = (\nu(p))_p$ for some smooth function $\nu : M \to S^2 \subset M_{3,1}(\mathbb{R})$ and where $|\vec{w}| = \sqrt{\vec{w} \cdot \vec{w}}$ for $\vec{w} \in M_{3,1}(\mathbb{R})$. This follows from the fact that the cross-product of two linearly independent vectors is orthogonal to the 2-plane spanned by the vectors. Recall that $\nu : M \to S^2$ is called the Gauss map of M and – by abusing language – $G : U \to S^2$ is sometimes also called Gauss map. Observe that

$$(\partial_1 F(q))_{F(q))} = F_*((\vec{e}_1)_q) \qquad \text{and} \qquad (\partial_2 F(q))_{F(q))} = F_*((\vec{e}_2)_q)$$

where $\{(\vec{e}_1)_q, (\vec{e}_2)_q\}$ denotes the standard basis of $T_q\mathbb{R}^2$. Suppressing base points to simplify notation and denoting the entries of $A(q)$ by $A_{ij}(q)$ for $1 \leqslant i, j \leqslant 2$, we have

$$\begin{aligned} A_{ij}(q) &= -\langle \mathrm{d}\nu(F_*(\vec{e}_i)), F_*(\vec{e}_j)\rangle = -\langle \mathrm{d}(\nu \circ F)(\vec{e}_i), F_*(\vec{e}_j)\rangle \\ &= -\langle \mathrm{d}G(\vec{e}_i), F_*(\vec{e}_j)\rangle, \end{aligned}$$

where the second equality follows from the chain rule and the third equality uses that $G = \nu \circ F$. Explicitly we thus have

$$\begin{aligned} A_{11}(q) &= -\partial_1 G(q) \cdot \partial_1 F(q), \\ A_{12}(q) &= -\partial_1 G(q) \cdot \partial_2 F(q) = -\partial_2 G(q) \cdot \partial_1 F(q) = A_{21}(q), \\ A_{22}(q) &= -\partial_2 G(q) \cdot \partial_2 F(q), \end{aligned}$$

where we write

$$\partial_1 G(q) := \frac{\partial G}{\partial u}(q) \qquad \text{and} \qquad \partial_2 G(q) := \frac{\partial G}{\partial v}(q)$$

Since $G(q) \cdot \partial_1 F(q) = 0$ for all $q \in U$, we have

$$\frac{\partial}{\partial u}(G \cdot F_u)(q) = 0 = \partial_1 G(q) \cdot \partial_1 F(q) + G(q) \cdot \partial^2_{11} F(q)$$

so that $\partial_1 G(q) \cdot \partial_1 F(q) = -G(q) \cdot \partial^2_{11} F(q)$, where we write

$$\partial^2_{11} F(q) := \frac{\partial^2 F}{\partial u \partial u}(q)$$

Using corresponding notation, we obtain likewise

$$\partial_2 G(q) \cdot \partial_2 F(q) = -G(q) \cdot \partial^2_{22} F(q)$$

and

$$\partial_1 G(q) \cdot \partial_2 F(q) = -G(q) \cdot \partial^2_{12} F(q) = -G(q) \cdot \partial^2_{21} F(q) = \partial_2 G(q) \cdot \partial_1 F(q).$$

In summary, we thus have

$$\boxed{\begin{aligned} A_{11}(q) &= G(q) \cdot \partial^2_{11} F(q), \\ A_{12}(q) &= G(q) \cdot \partial^2_{12} F(q) = G(q) \cdot \partial^2_{21} F(q) = A_{21}(q), \\ A_{22}(q) &= G(q) \cdot \partial^2_{22} F(q), \end{aligned}} \tag{3.11}$$

or written more succintly (while surpressing the base point)

$$A_{ij} = G \cdot \partial^2_{ij} F. \tag{3.12}$$

We next derive explicit identities for the functions $K \circ F : U \to \mathbb{R}$ and $H \circ F : U \to \mathbb{R}$. Recall that for all $q \in U$ the matrix $g(q)$ is the matrix representation with respect to the basis $\mathbf{b}_{F(q)}$ of the inner product $\langle \cdot, \cdot \rangle_{F(q)}$ on $T_{F(q)}\mathbb{R}^3$ restricted to $T_{F(p)}M$. From M06 Linear Algebra II we know that the restriction of an inner product to a subspace is non-degenerate. This is equivalent to $g(q)$ being an invertible matrix for all $q \in U$. We write $g^{-1} : U \to M_{2,2}(\mathbb{R})$ for the map which assign to a point $q \in U$ the inverse matrix of $g(q)$, that is, for all $q \in U$ we have $g^{-1}(q)g(q) = \mathbf{1}_2$, where $\mathbf{1}_2$ denotes the identity matrix of size 2.

Fix $q \in U$ and let $\mathbf{S}(q) \in M_{2,2}(\mathbb{R})$ denote the matrix representation of the shape operator $S_{F(q)}$ at $F(q)$ with respect to the ordered basis $\mathbf{b}_{F(q)}$ of $T_{F(q)}M$

$$\mathbf{S}(q) = \mathbf{M}(S_{F(q)}, \mathbf{b}_{F(q)})$$

Write $\mathbf{S}(q) = (S_{ij}(q))_{1\leqslant i,j\leqslant 2}$ for unique scalars $S_{ij}(q) \in \mathbb{R}$. Moreover let X_1, X_2 denote the basis vectors of the ordered basis $\mathbf{b}_{F(q)}$. Then we have

$$S_p(X_i) = \sum_{k=1}^{2} S_{ki}(q)X_k$$

Using this we compute

$$\begin{aligned} A_{ij}(q) &= -\langle S_p(X_i), X_j\rangle = -\left\langle \sum_{j=1}^{2} S_{ki}(q)X_k, X_j \right\rangle = -\sum_{k=1}^{2} S_{ki}(q)\langle X_k, X_j\rangle \\ &= -\sum_{k=1}^{2} S_{ki}(q)g_{kj}(q) = -\sum_{k=1}^{2} g_{jk}S_{ki}(q) = A_{ji}(q). \end{aligned}$$

In matrix notation we thus obtain the identity

$$A(q) = -g(q)\mathbf{S}(q) \qquad \Longleftrightarrow \qquad \mathbf{S}(q) = -g^{-1}(q)A(q). \tag{3.13}$$

Recall that the Gauss curvature at $p \in M$ is the determinant of S_p. Hence we conclude

$$K(F(q)) = \det \mathbf{S}(q) = \det\left(-g^{-1}(q)A(q)\right) = \det\left(g^{-1}(q)A(q)\right) = \frac{\det A(q)}{\det g(q)},$$

where the third equality uses that the determinant of a 2×2-matrix is unchanged when the matrix is multiplied by -1 and the last equality uses the product rule for the determinant.

For the mean curvature we obtain correspondingly $H(F(q)) = -\frac{1}{2}\operatorname{Tr}\left(g^{-1}(q)A(q)\right)$ so that in summary we have for all $q \in U$

$$\boxed{\begin{aligned} K(F(q)) &= \frac{\det A(q)}{\det g(q)}, \\ H(F(q)) &= -\frac{1}{2}\operatorname{Tr}\left(g^{-1}(q)A(q)\right). \end{aligned}} \tag{3.14}$$

Example 3.64 For the hyperbolic paraboloid with $F(q) = (u, v, uv)$ where $q = (u, v)$ we compute

$$G(q) = \frac{1}{\sqrt{1+u^2+v^2}}\begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix}.$$

From this one can calculate that

$$A(q) = \begin{pmatrix} 0 & \frac{1}{\sqrt{1+u^2+v^2}} \\ \frac{1}{\sqrt{1+u^2+v^2}} & 0 \end{pmatrix}$$

and Example 3.62 gives

$$g(q) = \begin{pmatrix} 1+v^2 & uv \\ uv & 1+u^2 \end{pmatrix}.$$

From this we obtain $\det g(q) = 1 + u^2 + v^2$ so that at $F(q) = (u, v, uv)$ we have Gauss curvature

$$K(F(q)) = -\frac{1}{(1+u^2+v^2)^2}$$

which is in agreement with (3.5). We also obtain

$$H(F(q)) = \frac{uv}{(1+u^2+v^2)^{3/2}},$$

which differs from (3.6) by a minus sign. This is no error however, since $G(q) = -\operatorname{grad} f(F(q))$ for all $q \in U$, where $f : \mathbb{R}^3 \to \mathbb{R}$, $p = (x, y, z) \mapsto xy - z$ is the defining function of the hyperbolic paraboloid.

Example 3.65 (Torus) For the torus we obtain

$$G(q) = \begin{pmatrix} \cos(u)\cos(v) \\ \cos(v)\sin(u) \\ \sin(v) \end{pmatrix}$$

and we can compute

$$A(q) = -\begin{pmatrix} \cos(v)(R + r\cos(v)) & 0 \\ 0 & r \end{pmatrix}$$

From which we deduce together with Exercise 3.63

$$K(F(q)) = \frac{\cos(v)}{r(R + r\cos(v))}$$

and

$$H(F(q)) = \frac{1}{2}\left(\frac{1}{r} + \frac{\cos(v)}{R + r\cos(v)}\right).$$

3.9 Immersed surfaces

All the calculations in the previous section also make sense if F is a smooth injective immersion. This motivates:

Definition 3.66 (Immersed surface) Let $U \subset \mathbb{R}^3$ be open and $F : U \to \mathbb{R}^3$ a smooth injective immersion. Then:

(i) the image $M := F(U) \subset \mathbb{R}^3$ is called an *immersed surface*;

(ii) the tangent space of M at $p = F(q)$ is defined as

$$T_{F(q)}M := \operatorname{span}\{(\partial_1 F(q))_{F(q)}, (\partial_2 F(q))_{F(q)}\}.$$

Remark 3.67

(i) In what follows, whenever we speak of a surface $M \subset \mathbb{R}^3$ we mean an immersed or embedded surface.

(ii) While we can define the tangent space at each point of an immersed surface, we have to be aware that immersed surfaces can have self intersections, compare with Figure 4.1.

(iii) The Gauss curvature, mean curvature, shape operator, first and second fundamental form and Gauss map are defined in terms of the expressions from the previous sections. Often, these quantities are interpreted as being defined on U. For instance, the Gauss curvature of an immersed surface is often interpreted as a function $K : U \to \mathbb{R}$.

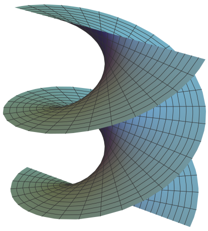

FIGURE 3.5. A (subset of the) helicoid.

(iv) We will occasionally also allow U to be non open, provided there exists an open subset $\tilde{U} \subset \mathbb{R}^2$ containing U and a smooth immersion $\tilde{F} : \tilde{U} \to \mathbb{R}^3$ so that the restriction of $\tilde{F}$ to $U \subset \tilde{U}$ is injective.

Example 3.68 (Helicoid) Consider

$$F : \mathbb{R}^2 \to \mathbb{R}^3, \qquad q = (u, v) \mapsto F(q) = (u\cos(v), u\sin(v), v)$$

Clearly, F is smooth and injective and a calculation shows that F is an immersion, hence $M = F(\mathbb{R}^2) \subset \mathbb{R}^3$ is an immersed surface called the *Helicoid*. Here we compute

$$g(q) = \begin{pmatrix} 1 & 0 \\ 0 & 1+u^2 \end{pmatrix} \qquad \text{and} \qquad G(q) = \frac{1}{\sqrt{1+u^2}} \begin{pmatrix} \sin(v) \\ -\cos(v) \\ u \end{pmatrix}$$

as well as

$$A(q) = \begin{pmatrix} 0 & -\frac{1}{\sqrt{1+u^2}} \\ -\frac{1}{\sqrt{1+u^2}} & 0 \end{pmatrix}.$$

Which gives

$$K(q) = -\frac{1}{(1+u^2)^2} \qquad \text{and} \qquad H(q) = 0.$$

The mean curvature of a Helicoid is identically 0. Such surfaces are called minimal surfaces.

Definition 3.69 (Minimal surface) An immersed or embedded surface $M \subset \mathbb{R}^3$ whose mean curvature is identically 0 is called a *minimal surface*.

Remark 3.70 Minimal surfaces are mathematical idealisations of soap films and belong to the most intensively studied surfaces in geometry. Despite having mathematical origins that date back to the 18th century, they are still actively studied.

An interesting class of immersed surfaces arises from rotating a curve in the xz-plane around the z-axis.

Example 3.71 (Surface of revolution) Let I be an interval and $\gamma = (\gamma_1, \gamma_2) : I \to \mathbb{R}^2$ a smooth injective immersed curve with $\gamma_1(t) > 0$ for all $t \in I$. Consider $F : [0, 2\pi) \times I \to \mathbb{R}^3$ defined by

$$(u, v) \mapsto f_{\mathbf{R}_u}(\gamma_1(v), 0, \gamma_2(v)) = (\gamma_1(v)\cos(u), \gamma_1(v)\sin(u), \gamma_2(v))$$

where $\mathbf{R}_u$ is the matrix corresponding to rotation around the z-axis with angle u

$$\mathbf{R}_u = \begin{pmatrix} \cos(u) & -\sin(u) & 0 \\ \sin(u) & \cos(u) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, one can easily check that $M = \operatorname{Im}(F) \subset \mathbb{R}^3$ is an immersed surface known as a *surface of revolution*. We compute the Gauss and mean curvature in the case where γ is a unit speed curve. We have

$$\partial_1 F(q) = \begin{pmatrix} -\gamma_1(v)\sin(u) \\ \gamma_1(v)\cos(u) \\ 0 \end{pmatrix} \quad \text{and} \quad \partial_2 F(q) = \begin{pmatrix} \gamma_1'(v)\cos(u) \\ \gamma_1'(v)\sin(u) \\ \gamma_2'(v) \end{pmatrix}.$$

from which we compute

$$g(q) = \begin{pmatrix} \gamma_1(v)^2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad G(q) = \begin{pmatrix} \cos(u)\gamma_2'(v) \\ \sin(u)\gamma_2'(v) \\ -\gamma_1'(v) \end{pmatrix}$$

as well as

$$A(q) = \begin{pmatrix} -\gamma_1(v)\gamma_2'(v) & 0 \\ 0 & \gamma_1''(v)\gamma_2'(v) - \gamma_1'(v)\gamma_2''(v) \end{pmatrix}.$$

Hence we obtain

$$K(q) = \frac{\gamma_2'(v)\left(\gamma_1'(v)\gamma_2''(v) - \gamma_1''(v)\gamma_2'(v)\right)}{\gamma_1(v)}$$

Differentiating

$$\gamma_1'(v)^2 + \gamma_2'(v)^2 = 1$$

with respect to v we deduce

$$\gamma_1'(v)\gamma_1''(v) = -\gamma_2'(v)\gamma_2''(v),$$

so that

$$K(q) = \frac{-\gamma_1''(v)\gamma_1'(v)^2 - \gamma_1''(v)\gamma_2'(v)^2}{\gamma_1(v)} = -\frac{\gamma_1''(v)}{\gamma_1(v)}.$$

For the mean curvature we obtain

$$H(q) = \frac{1}{2}\left(\frac{\gamma_2'(v)}{\gamma_1(v)} + \gamma_1'(v)\gamma_2''(v) - \gamma_2'(v)\gamma_1''(v)\right).$$

Notice that if $\kappa : I \to \mathbb{R}$ denotes the signed curvature of the plane curve $\gamma = (\gamma_1, \gamma_2) : I \to \mathbb{R}^2$, then we can write

$$H(q) = \frac{1}{2}\left(\kappa(v) + \frac{\gamma_2'(v)}{\gamma_1(v)}\right).$$

Exercise 3.72 (Catenoid) The surface of revolution arising from $\gamma_1(v) = \cosh(v)$ and $\gamma_2(v) = v$ is known as the *Catenoid*. Show that the Catenoid has mean curvature identical to 0. Warning: The formula for H from Example 3.71 cannot be used, since $\gamma = (\gamma_1, \gamma_2) : \mathbb{R} \to \mathbb{R}^2$ is not a unit speed curve.

Remark 3.73 The Catenoid is the first non-trivial example of a minimal surface (the plane is a trivial example). It was discovered in 1744 by the Swiss Mathematician Leonard Euler.

Exercise 3.74 Show that the surface of revolution arising from the tractrix – known as the *pseudo-sphere* – has constant negative Gauss curvature.

CHAPTER 4

Intrinsic surface geometry

WEEK 7

An important observation in geometry is that the Gauss curvature of an embedded surface can be expressed in terms of the first fundamental form only. Gauss, who discovered this fact, was so astonished by it that he called it a "Theorema Egregium" (Latin for "Remarkable Theorem"). Geometric quantities associated to a surface which can be obtained from computing inner products between tangent vectors – that is, quantities that are computable once we know the first fundamental form – are called *intrinsic*. Intrinsic quantities are in contrast to *extrinsic* quantities which cannot be computed from knowing the first fundamental form alone. Prototypical examples of extrinsic quantities associated to an embedded surface $M \subset \mathbb{R}^3$ are the second fundamental form, the mean curvature and the unit normal field. The intuition for intrinsic vs extrinsic is that intrinsic quantities do not rely on the ambient space $\mathbb{R}^3$ in which the surface is embedded, whereas extrinsic quantities do.

Recall that the Gauss curvature is the product of the principal curvatures which can be computed as the signed curvature of the curve cut out of the surface by intersecting it with a suitable affine 2-plane. At $p \in M$ the affine 2-plane is spanned by a unit normal vector $N(p)$ and a tangent vector $\vec{v}_p$. The unit normal vector $N(p)$ being an extrinsic quantity, it is not clear at all that the Gauss curvature can be expressed without involving $N(p)$, this is however the case as we will see below. Gauss' Theorema Egregium lead mathematicians to consider geometric spaces which are not necessarily embedded in a surrounding ambient space such as $\mathbb{R}^3$. This point of view is relevant in particular in physics. As far as we know our universe does not sit inside a *larger ambient universe*, but is a *geometric space in itself*.

4.1 The Gauss–Codazzi equations

Specifying an immersed surface $F : U \to \mathbb{R}^3$ involves choosing 3 functions $F_1, F_2, F_3 : U \to \mathbb{R}$. Recall that to an immersed surface $F : U \to \mathbb{R}^3$ we associated two maps $g, A : U \to M_{2,2}(\mathbb{R})$ taking values in the symmetric 2×2-matrices. The map g encodes the first fundamental form and the map A encodes the second fundamental form. We can change our view point and *prescribe* two matrix-valued maps g, A on an open set $U \subset \mathbb{R}^2$ and ask whether g and A arise from an immersion $F : U \to \mathbb{R}^3$ via the expressions given in (3.8) and (3.11). Thinking of g_{ij}, A_{ij} for $1 \leqslant i, j \leqslant 2$ as given and F_1, F_2, F_3 as *unknown functions*, the equations (3.8) and (3.11) are a *system of partial differential equations*. Partial differential equations are the (generally speaking more complicated) counterparts to *ordinary differential equations*, the key difference being that the sought after functions are allowed to depend on more than one variable. Many important laws of nature can be phrased as partial differential equations, in particular, the so-called *Einstein field equations* describing gravity, the *Schrödinger equation* arising in quantum mechanics and the *Maxwell equations* governing the laws of electromagnetism. Understanding partial differential equations is a fundamental part of modern mathematics.

The two systems (3.8) and (3.11) give us 6 equations for 3 unknown functions F_1, F_2, F_3. Roughly speaking, whenever we have more equations (here 6) than unknowns (here 3) we should expect some *compatibility conditions* among the equations so that we can find any solutions. For historical reasons, in the theory of partial differential equations compatibility conditions are often called *integrability conditions*.

In what follows we derive such conditions. That is, we derive conditions for the functions g_{ij}, A_{ij} (and their derivatives) that are necessary for (3.8) and (3.11) to have a solution F_1, F_2, F_3. They appear in Proposition 4.5 below.

Let $F : U \to \mathbb{R}^3$ be an immersed surface so that

$$\mathbf{c}(q) := (\partial_1 F(q), \partial_2 F(q), G(q))$$

is an ordered basis of $M_{3,1}(\mathbb{R})$ for all $q \in U$. For all $q \in U$ and all $1 \leqslant i, j \leqslant 2$, the vector $\partial^2_{ij} F(q) \in M_{3,1}(\mathbb{R})$ can thus be written as a linear combination of the elements of $\mathbf{c}(q)$. We can therefore find unique functions $\Gamma^k_{ij} : U \to \mathbb{R}$ and $B_{ij} : U \to \mathbb{R}$ for $1 \leqslant i, j, k \leqslant 2$ so that

$$\partial^2_{ij} F(q) = \Gamma^1_{ij}(q)\partial_1 F(q) + \Gamma^2_{ij}(q)\partial_2 F(q) + B_{ij}(q)G(q). \tag{4.1}$$

Taking the inner product with G we obtain

$$A_{ij} = G \cdot \partial^2_{ij} F = \Gamma^1_{ij} (G \cdot \partial_1 F) + \Gamma^2_{ij} (G \cdot \partial_2 F) + B_{ij} (G \cdot G) = B_{ij},$$

where we suppress the base point, we use (3.12), $G(q) \cdot G(q) = 1$ and that $G(q)$ is orthogonal to $\partial_1 F(q)$ and to $\partial_2 F(q)$. Taking inner products with $\partial_1 F$ and $\partial_2 F$ we obtain

$$\partial_1 F \cdot \partial^2_{ij} F = \Gamma^1_{ij} (\partial_1 F \cdot \partial_1 F) + \Gamma^2_{ij} (\partial_1 F \cdot \partial_2 F) = \Gamma^1_{ij} g_{11} + \Gamma^2_{ij} g_{12}$$

and

$$\partial_2 F \cdot \partial^2_{ij} F = \Gamma^1_{ij} (\partial_2 F \cdot \partial_1 F) + \Gamma^2_{ij} (\partial_2 F \cdot \partial_2 F) = \Gamma^1_{ij} g_{21} + \Gamma^2_{ij} g_{22},$$

where we use (3.8).

Remark 4.1 (Einstein Summation convention) In what follows we employ a useful notational convention going back to A. Einstein. Whenever an index appears as an upper index as well as a lower index in the same term, then it is automatically summed over. For instance, in the expression $\Gamma^l_{ij} g_{kl}$ the index l occurs both as an upper index and a lower index, hence we have

$$\Gamma^l_{ij} g_{kl} = \Gamma^1_{ij} g_{k1} + \Gamma^2_{ij} g_{k2}.$$

Using the Einstein summation convention, the above equations can be written as

$$\partial_k F \cdot \partial^2_{ij} F = \Gamma^l_{ij} g_{kl}.$$

Now notice that for $1 \leqslant i, j, k \leqslant 2$ we have

$$\partial_i g_{jk} = \partial_i (\partial_j F \cdot \partial_k F) = \partial^2_{ij} F \cdot \partial_k F + \partial_j F \cdot \partial^2_{ik} F.$$

From this we compute

$$\begin{aligned}\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} = \partial^2_{ij} F \cdot \partial_k F + \partial_j F \cdot \partial^2_{ik} F + \partial^2_{ji} F \cdot \partial_k F + \partial_i F \cdot \partial^2_{jk} F \\ - \partial^2_{ki} F \cdot \partial_j F - \partial_i F \cdot \partial^2_{kj} F\end{aligned}$$

so that

$$\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} = 2\partial_k F \cdot \partial^2_{ij} F,$$

where we use that $\partial^2_{ij} F = \partial^2_{ji} F$. In summary we have

$$\Gamma^l_{ij} g_{kl} = \frac{1}{2} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}). \tag{4.2}$$

Recall that we write $g^{-1} : U \to M_{2,2}(\mathbb{R})$ for the map which assigns to $q \in U$ the inverse of the matrix $g(q)$. It is customary to write

$$g^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}$$

for functions $g^{ij} : U \to \mathbb{R}$ which satisfy $g^{12} = g^{21}$. By definition, we have

$$g^{rk} g_{kl} = g^{r1} g_{1l} + g^{r2} g_{2l} = \begin{cases} 1, & r = l, \\ 0, & r \neq l. \end{cases}$$

Using this we can compute

$$g^{rk} \Gamma^l_{ij} g_{kl} = \Gamma^l_{ij} g^{rk} g_{kl} = \Gamma^r_{ij}.$$

Finally, using (4.2) we thus have

$$\Gamma^r_{ij} = \frac{1}{2} g^{rk} \left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \right).$$

Definition 4.2 (Christoffel symbols) The functions $\Gamma^l_{ij} : U \to \mathbb{R}$ defined by

$$\Gamma^l_{ij} = \frac{1}{2} g^{lk} \left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \right) \qquad 1 \leqslant i, j, k, l \leqslant 2.$$

are the *Christoffel symbols* associated to the immersion $F : U \to \mathbb{R}^3$.

Notice that the Christoffel symbols satisfy

$$\Gamma^l_{ij} = \Gamma^l_{ji}.$$

Example 4.3 (Example 3.71 continued) For a surface of revolution, we computed that

$$g(q) = \begin{pmatrix} \gamma_1(v)^2 & 0 \\ 0 & 1 \end{pmatrix}$$

so that

$$g^{-1}(q) = \begin{pmatrix} \frac{1}{\gamma_1(v)^2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover, we have

$$\partial_2 g_{11}(q) = 2\gamma_1(v)\gamma_1'(v)$$

and $\partial_i g_{jk} = 0$ otherwise. It follows that

$$\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^2_{22} = 0$$

and

$$\begin{aligned} \Gamma^1_{12}(q) = \Gamma^1_{21}(q) &= \frac{1}{2} g^{11}(q) \left(\partial_1 g_{12}(q) + \partial_2 g_{11}(q) - \partial_1 g_{12}(q) \right) \\ &= \frac{1}{2} g^{11}(q) \partial_2 g_{11}(q) = \frac{2\gamma_1(v)\gamma_1'(v)}{2\gamma_1(v)^2} = \frac{\gamma_1'(v)}{\gamma_1(v)} \end{aligned}$$

as well as

$$\begin{aligned} \Gamma^2_{11}(q) &= \frac{1}{2} g^{22}(q) \left(\partial_1 g_{12}(q) + \partial_1 g_{12}(q) - \partial_2 g_{11}(q) \right) \\ &= -\frac{1}{2} g^{22}(q) \partial_2 g_{11}(q) = -\frac{1}{2} 2\gamma_1(v)\gamma_1'(v) = -\gamma_1(v)\gamma_1'(v). \end{aligned}$$

Example 4.4 (Example 3.68 continued) For the Helicoid we compute that

$$g(q) = \begin{pmatrix} 1 & 0 \\ 0 & 1 + u^2 \end{pmatrix}$$

so that

$$g^{-1}(q) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1+u^2} \end{pmatrix}.$$

Moreover, we have

$$\partial_1 g_{22}(q) = 2u$$

and $\partial_i g_{jk}$ otherwise. It follows that

$$\Gamma^2_{22} = \Gamma^2_{11} = \Gamma^1_{21} = \Gamma^1_{12} = \Gamma^1_{11} = 0$$

and

$$\begin{aligned} \Gamma^2_{12}(q) = \Gamma^2_{21}(q) &= \frac{1}{2} g^{22}(q) \left(\partial_1 g_{22}(q) + \partial_2 g_{12}(q) - \partial_2 g_{12}(q)\right)) \\ &= \frac{1}{2} g^{22}(q) \partial_1 g_{22}(q) = \frac{2u}{2(1+u^2)} = \frac{u}{1+u^2}, \end{aligned}$$

as well as

$$\begin{aligned} \Gamma^1_{22} &= \frac{1}{2} g^{11}(q) \left(2\partial_2 g_{12}(q) - \partial_1 g_{22}(q)\right) = -\frac{1}{2} g^{11}(q) \partial_1 g_{22}(q) \\ &= -\frac{1}{2} 2u = -u. \end{aligned}$$

We now return to our problem of determining integrability conditions for finding $F : U \to \mathbb{R}^3$ when we are given the functions g_{ij}, A_{ij} on U. Using the summation convention, we can write (4.1) as

$$\partial^2_{jk} F = A_{jk} G + \Gamma^l_{jk} \partial_l F.$$

Using this we compute

$$\begin{aligned} Q_{jikm} :&= \left[\partial_i \left(\partial^2_{jk} F\right)\right] \cdot \partial_m F = \left[\partial_i \left(A_{jk} G + \Gamma^l_{jk} \partial_l F\right)\right] \cdot \partial_m F \\ &= \partial_i A_{jk} \underbrace{G \cdot \partial_m F}_{=0} + A_{jk} \underbrace{\partial_i G \cdot \partial_m F}_{=-A_{im}} + \partial_i \Gamma^l_{jk} \underbrace{\partial_l F \cdot \partial_m F}_{=g_{lm}} + \Gamma^l_{jk} \underbrace{\partial^2_{il} F \cdot \partial_m F}_{=\Gamma^r_{il} g_{mr}} \\ &= -A_{jk} A_{im} + \partial_i \Gamma^l_{jk} g_{lm} + \Gamma^l_{jk} \Gamma^r_{il} g_{mr}. \end{aligned}$$

Indices that are summed over can be given new “names”, so that

$$\Gamma^l_{jk} \Gamma^r_{il} g_{mr} = \Gamma^a_{jk} \Gamma^r_{ia} g_{mr} = \Gamma^a_{jk} \Gamma^b_{ia} g_{mb} = \Gamma^r_{jk} \Gamma^b_{ir} g_{mb} = \Gamma^r_{jk} \Gamma^l_{ir} g_{ml}.$$

Since $g_{ml} = g_{lm}$ we thus obtain

$$Q_{jikm} = -A_{jk} A_{im} + \left(\partial_i \Gamma^l_{jk} + \Gamma^r_{jk} \Gamma^l_{ir}\right) g_{lm}.$$

Using that third derivatives commute, we have $\partial_j(\partial^2_{ik} F) = \partial_i(\partial^2_{jk} F)$ and hence

$$0 = Q_{ijkm} - Q_{jikm} = \left(\partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk} + \Gamma^r_{ik} \Gamma^l_{jr} - \Gamma^r_{jk} \Gamma^l_{ir}\right) g_{lm} - \left(A_{ik} A_{jm} - A_{jk} A_{im}\right).$$

Writing

$$R_{ijkm} = \left(\partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk} + \Gamma^r_{ik} \Gamma^l_{jr} - \Gamma^r_{jk} \Gamma^l_{ir}\right) g_{lm}, \tag{4.3}$$

we have the so-called *Gauss equations*

$$R_{ijkm} = A_{ik} A_{jm} - A_{jk} A_{im}, \tag{4.4}$$

which must hold for all $1 \leqslant i, j, k, m \leqslant 2$. The functions R_{ijkm} depend on the g_{lm} and the Christoffel symbols only, thus they can be computed from knowing the functions g_{ij}. If we are given functions g_{ij}, A_{ij} on U, then the Gauss equations are necessary conditions for the existence of an immersion $F : U \to \mathbb{R}^3$ realising g_{ij}, A_{ij}. We can derive more necessary conditions as follows: Consider

$$\begin{aligned} P_{jik} :&= \left[\partial_i \left(\partial^2_{jk} F\right)\right] \cdot G = \left[\partial_i \left(A_{jk} G + \Gamma^l_{jk} \partial_l F\right)\right] \cdot G \\ &= \partial_i A_{jk} \underbrace{G \cdot G}_{=1} + A_{jk} \partial_i G \cdot G + \partial_i \Gamma^l_{jk} \underbrace{\partial_l F \cdot G}_{=0} + \Gamma^l_{jk} \partial^2_{il} F \cdot G. \end{aligned} \tag{4.5}$$

Since $G(q) \cdot G(q) = 1$, it follows as before that $\partial_i G(q) \cdot G(q) = 0$ for all $q \in U$ and $i = 1, 2$. Therefore (4.5) together with (4.1) gives

$$P_{jik} = \partial_i A_{jk} + \Gamma^l_{jk}(\Gamma^m_{il}\partial_m F + A_{il}G) \cdot G = \partial_i A_{jk} + \Gamma^l_{jk} A_{il}.$$

Again, using that third partial derivatives commute, we arrive at

$$0 = P_{ijk} - P_{jik} = \partial_j A_{ik} - \partial_i A_{jk} + \Gamma^l_{ik} A_{jl} - \Gamma^l_{jk} A_{il}.$$

Equivalently, at the so-called *Codazzi equations*

$$\partial_j A_{ik} - \partial_i A_{jk} = \Gamma^l_{jk} A_{il} - \Gamma^l_{ik} A_{jl}, \tag{4.6}$$

which must hold for all $1 \leqslant i, j, k \leqslant 2$. This shows:

Proposition 4.5 (Gauss–Codazzi equations) *Let $U \subset \mathbb{R}^2$ be an open subset and $g_{ij}, A_{ij} : U \to \mathbb{R}$ smooth functions for $1 \leqslant i, j \leqslant 2$ with $g_{12} = g_{21}$ and $A_{12} = A_{21}$. Then the Gauss – and Codazzi equations*

$$R_{ijkm} = A_{ik}A_{jm} - A_{jk}A_{im} \qquad \text{and} \qquad \partial_j A_{ik} - \partial_i A_{jk} = \Gamma^l_{jk} A_{il} - \Gamma^l_{ik} A_{jl}, \tag{4.7}$$

are necessary conditions for the existence of a smooth immersion $F : U \to \mathbb{R}^3$ whose associated functions via (3.9) and (3.12) are g_{ij} and A_{ij}.

Remark 4.6

(i) A theorem which goes beyond the scope of this course states that if U is so-called *simply connected* (which is in particular the case if U is a rectangle), then the equations (4.7) are also sufficient, provided $g_{ij}(q)$ is positive definite for all $q \in U$. That is, if g_{ij}, A_{ij} are given functions on U satisfying (4.7) and g_{ij} is positive definite, then there exists an immersion $F : U \to \mathbb{R}^3$ realising g_{ij} and A_{ij} and moreover, the image of this immersion if unique up to post composition by a Euclidean motion.

(ii) Loosely speaking this all states that the functions g_{ij} and A_{ij} capture an immersed surface up to Euclidean motion.

(iii) In the case of a curve in $\mathbb{R}^2$, we saw that the signed curvature captures the curve up to Euclidean motion. We can prescribe any smooth function as the signed curvature of a plane curve, whereas in the case of a surface the functions g_{ij}, A_{ij} that we prescribe must satisfy the *integrability conditions* (4.7).

4.2 The covariant derivative revisited

Recall that

$$\begin{pmatrix} g_{11}(q) & g_{12}(q) \\ g_{12}(q) & g_{22}(q) \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} A_{11}(q) & A_{12}(q) \\ A_{12}(q) & A_{22}(q) \end{pmatrix}$$

are the matrix representations of the first – and second – fundamental form $\mathrm{I}_{F(q)}, \mathrm{II}_{F(q)}$ at $F(q)$ with respect to the ordered basis $((\partial_1 F(q))_{F(q)}, (\partial_2 F(q))_{F(q)})$ of $T_{F(q)}M$, respectively. It is natural to wonder whether the Christoffel symbols Γ^i_{jk} also encode a natural map.

Recall that if $M \subset \mathbb{R}^3$ is an embedded surface and $\gamma : I \to M$ a smooth curve and $X : I \to TM$ a vector field along γ, so that $X(t) \in T_{\gamma(t)}M$, then we defined the covariant derivative of X as

$$\frac{\mathrm{D}X}{\mathrm{d}t}(t) = \Pi^{\perp}_{T_{\gamma(t)}M}(\dot{X}(t))$$

for all $t \in I$. Phrased differently, $\frac{\mathrm{D}X}{\mathrm{d}t}(t)$ is the tangential component of the vector $\dot{X}(t) \in T_{\gamma(t)}M$ with respect to the direct sum decomposition

$$T_{\gamma(t)}\mathbb{R}^3 = T_{\gamma(t)}M \oplus T_{\gamma(t)}M^{\perp}.$$

We can use the covariant derivative of a vector field along a curve to define a *directional derivative of a vector field*. If $Y : M \to TM$ is a smooth vector field on M and $\vec{v}_p \in TM$, we define the derivative of the vector field Y in the tangent direction $\vec{v}_p$ by

$$\nabla_{\vec{v}_p} Y := \frac{\mathrm{D}}{\mathrm{d}t}(Y \circ \gamma)(0),$$

where $\gamma : (-\epsilon, \epsilon) \to M$ is a smooth curve with $\gamma(0) = p$, $\dot{\gamma}(0) = \vec{v}_p$ and $\epsilon > 0$. We have to make sure that the choice of γ does not matter, that is $\nabla_{\vec{v}_p} Y$ does only depend on Y and $\vec{v}_p$. Write

$$Y(p) = \begin{pmatrix} Y^1(p) \\ Y^2(p) \\ Y^3(p) \end{pmatrix}_p$$

for smooth functions $Y^i : M \to \mathbb{R}$. Then by definition

$$\nabla_{\vec{v}_p} Y = \Pi^{\perp}_{T_pM}(\dot{Z}(0)),$$

where $Z = Y \circ \gamma$. We have

$$\dot{Z}(0) = \begin{pmatrix} (Y^1 \circ \gamma)'(0) \\ (Y^2 \circ \gamma)'(0) \\ (Y^3 \circ \gamma)'(0) \end{pmatrix}_{\gamma(0)} = \begin{pmatrix} \mathrm{d}Y^1(\dot{\gamma}(0)) \\ \mathrm{d}Y^2(\dot{\gamma}(0)) \\ \mathrm{d}Y^3(\dot{\gamma}(0)) \end{pmatrix}_{\gamma(0)} = \begin{pmatrix} \mathrm{d}Y^1(\vec{v}_p) \\ \mathrm{d}Y^2(\vec{v}_p) \\ \mathrm{d}Y^3(\vec{v}_p) \end{pmatrix}_p.$$

We conclude that $\dot{Z}(0)$ does only depend on Y and $\vec{v}_p$, and hence so does $\nabla_{\vec{v}_p} Y$, since $\Pi^{\perp}_{T_pM}$ does not depend on the choice of curve γ.

Let $\mathfrak{X}(M)$ denote the set of smooth vector fields on M. We define addition in the natural way, that is, for $X, Y \in \mathfrak{X}(M)$, we define for all $p \in M$

$$(X + Y)(p) := X(p) + Y(p).$$

Moreover, for a smooth function $f : M \to \mathbb{R}$ we define

$$(fX)(p) := f(p)X(p).$$

For two vector fields $X, Y \in \mathfrak{X}(M)$ and all $p \in M$, we define

$$(\nabla_X Y)(p) := \nabla_{X(p)} Y \in T_pM.$$

With these rules in place we can think of ∇ as a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$(X, Y) \mapsto \nabla_X Y$$

The map ∇ is also called the *covariant derivative*.

Lemma 4.7 *Let $X_1, X_2, Y_1, Y_2 \in \mathfrak{X}(M)$ and $f : M \to \mathbb{R}$ a smooth function. Then the covariant derivative satisfies:*

(i) $\nabla_{X_1+X_2} Y_1 = \nabla_{X_1} Y_1 + \nabla_{X_2} Y_1$;
(ii) $\nabla_{X_1}(Y_1 + Y_2) = \nabla_{X_1} Y_1 + \nabla_{X_2} Y_2$;
(iii) $\nabla_{fX_1} Y_1 = f\nabla_{X_1} Y_1$;
(iv) $\nabla_{X_1}(fY_1) = f\nabla_{X_1} Y_1 + \mathrm{d}f(X_1)Y_1$.

Proof Exercise. □

Remark 4.8 In Lemma 4.7, $\mathrm{d}f(X_1)$ is the smooth function on M defined by

$$\mathrm{d}f(X_1) : M \to \mathbb{R}, \qquad p \mapsto \mathrm{d}f(X_1(p)).$$

By construction the covariant derivative ∇ does depend on the first fundamental form only, it is thus an object of intrinsic surface geometry. In fact, the Christoffel symbols do encode ∇, more precisely, we have the following statement:

Proposition 4.9 *Let $M \subset \mathbb{R}^3$ be a surface and $F : U \to M$ a local parametrisation of M with Christoffel symbols $\Gamma^k_{ij} : U \to \mathbb{R}$ for $i, j, k = 1, 2$. Then on $\mathrm{Im}(F) \subset M$ we obtain vector fields B_i for $i = 1, 2$ defined by the rule*

$$B_i(F(q)) = (\partial_i F(q))_{F(q)}$$

for all $q \in U$. For these vector fields we have

$$(\nabla_{B_i} B_j)(F(q)) = \Gamma^k_{ij}(q) B_k(F(q))$$

for all $q \in U$ and where we employ the summation convention.

Remark 4.10 Since $F : U \to M$ is an immersion, it follows that $\{B_1(p), B_2(p)\}$ is a basis of T_pM for all $p \in F(U)$.

For the proof we need the following:

Lemma 4.11 *Let $M \subset \mathbb{R}^3$ be a surface and $F : U \to M$ a local parametrisation of M. Suppose $c = (c^1, c^2) : I \to U$ is a smooth curve and $X : I \to M$ is a vector field along the curve $\gamma = F \circ c : I \to M$. Writing*

$$X(t) = X^j(t) B_j(\gamma(t)) \tag{4.8}$$

for unique smooth functions $X^i : I \to \mathbb{R}$, we have

$$\frac{\mathrm{D}X}{\mathrm{d}t}(t) = \left(\frac{\mathrm{d}X^l}{\mathrm{d}t}(t) + \Gamma^l_{ij}(c(t)) X^j(t) \frac{\mathrm{d}c^i}{\mathrm{d}t}(t) \right) B_l(\gamma(t)), \tag{4.9}$$

where in (4.8) and (4.9) we employ the summation convention.

Proof Taking the time derivative of (4.8), we obtain

$$\dot{X}(t) = \left(\frac{\mathrm{d}X^j}{\mathrm{d}t}(t) \partial_j F(c(t)) + X^j(t) \frac{\mathrm{d}c^j}{\mathrm{d}t}(t) \partial^2_{ij} F(c(t)) \right)_{\gamma(t)}.$$

Since $\partial^2_{ji} F = \Gamma^l_{ji} \partial_l F + A_{ji} G$, we get

$$\dot{X}(t) = \left(\frac{\mathrm{d}X^j}{\mathrm{d}t}(t) \partial_j F(c(t)) \right.$$
$$\left. + X^j(t) \frac{\mathrm{d}c^i}{\mathrm{d}t}(t) \left(\Gamma^l_{ij}(c(t)) \partial_l F(c(t)) + A_{ij}(c(t)) G(c(t)) \right) \right)_{\gamma(t)}$$

The tangential component of $\dot{X}(t)$ is thus given by

$$\frac{\mathrm{D}X}{\mathrm{d}t}(t) = \left(\frac{\mathrm{d}X^l}{\mathrm{d}t}(t) + \Gamma^l_{ij}(c(t)) X^j(t) \frac{\mathrm{d}c^i}{\mathrm{d}t}(t) \right) B_l(\gamma(t)),$$

as claimed. □

Proof of Proposition 4.9 In order to compute $(\nabla_{B_i} B_j)(F(q))$ we can choose a curve $\gamma : (-\epsilon, \epsilon) \to M$ so that $\dot\gamma(0) = B_i(F(q))$ and then evaluate $\frac{\mathrm{D}}{\mathrm{d}t} X(0)$ using the formula (4.9) for $X := B_j \circ \gamma$. Let e_1, e_2 denote the standard basis of $\mathbb{R}^2$, interpreted as points, and consider the curve $c : (-\epsilon, \epsilon) \to U$ defined by the rule $c(t) = q + te_i$ for ϵ sufficiently small. Notice that $\gamma = F \circ c$ satisfies $\gamma(0) = F(q)$ and moreover $\dot\gamma(0) = B_i(F(q))$. Since

$$X(t) = B_j(\gamma(t)) = \Big(\partial_j F(c(t))\Big)_{\gamma(t)}$$

it follows that the functions X^i in (4.8) are given by $X^i(t) = \delta^i_j$ for all $t \in (-\epsilon, \epsilon)$. Moreover, we also have

$$\frac{\mathrm{d}c^k}{\mathrm{d}t}(t) = \delta^k_i.$$

Renaming indices in (4.9) and evaluating at $t = 0$ we obtain

$$\begin{aligned}(\nabla_{B_i} B_j)(F(q)) = \frac{\mathrm{D}X}{\mathrm{d}t}(0) &= \left(\frac{\mathrm{d}X^l}{\mathrm{d}t}(0) + \Gamma^l_{rm}(c(0))X^m(0)\frac{\mathrm{d}c^r}{\mathrm{d}t}(0)\right)\Big(\partial_l F(c(0))\Big)_{\gamma(0)}\\ &= \left(\Gamma^l_{rm}(q)\delta^m_j\delta^r_i\right)\Big(\partial_l F(q)\Big)_{F(q)} = \Gamma^l_{ij}(q)B_l(F(q)),\end{aligned}$$

as claimed. ☐

Recall that a geodesic $\gamma : I \to M$ must satisfy $\frac{\mathrm{D}\dot\gamma}{\mathrm{d}t}(t) = 0$ for all $t \in I$. From Proposition 4.9 we thus obtain:

Corollary 4.12 *Let $M \subset \mathbb{R}^3$ be a surface and $F : U \to M$ a local parametrisation of M. Suppose $c = (c^1, c^2) : I \to U$ is a smooth curve. Then $\gamma = F \circ c : I \to M$ is a geodesic if and only if c satisfies the so-called geodesic equation*

$$\frac{\mathrm{d}^2 c^l}{\mathrm{d}t^2}(t) + \Gamma^l_{ij}(c(t))\frac{\mathrm{d}c^i}{\mathrm{d}t}(t)\frac{\mathrm{d}c^j}{\mathrm{d}t}(t) = 0 \tag{4.10}$$

for all $t \in I$ and where we employ the summation convention.

Proof This follows immediately from (4.9) for $X = \dot\gamma$ so that $X^i(t) = \frac{\mathrm{d}c^i}{\mathrm{d}t}(t)$. ☐

Also, for vector fields $X, Y \in \mathfrak{X}(M)$ we obtain a function

$$\langle X, Y\rangle : M \to \mathbb{R}, \qquad p \mapsto \langle X(p), Y(p)\rangle_p$$

and with this definition we have for all $Z \in \mathfrak{X}(M)$

$$\mathrm{d}\left(\langle X, Y\rangle\right)(Z) = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle, \tag{4.11}$$

which can again be deduced from (4.9).

Example 4.13 (Geodesics on the helicoid – Example 4.4 continued) For explicit calculations it is often convenient to write $u(t)$ instead of $c^1(t)$ and $v(t)$ instead of $c^2(t)$. Doing so we obtain for the geodesic equation on the helicoid

$$\begin{aligned}0 &= \frac{\mathrm{d}^2 u}{\mathrm{d}t^2}(t) - u(t)\frac{\mathrm{d}v}{\mathrm{d}t}(t)\frac{\mathrm{d}v}{\mathrm{d}t}(t),\\ 0 &= \frac{\mathrm{d}^2 v}{\mathrm{d}t^2}(t) + \frac{2u(t)}{1+u(t)^2}\frac{\mathrm{d}u}{\mathrm{d}t}(t)\frac{\mathrm{d}v}{\mathrm{d}t}(t)\end{aligned}$$

or, using primes to indicate derivatives and omitting writing t, we have

$$u'' = uv'v' \qquad \text{and} \qquad v'' = -\frac{2u}{1+u^2}u'v'.$$

Example 4.14 (Geodesics on the torus) We can parametrise the torus in terms of

$$F : (0, 2\pi) \times (0, 2\pi r), \qquad (u, v) \mapsto (\cos(u)\gamma_1(v), \sin(u)\gamma_1(v), \gamma_2(v))$$

where $\gamma_1(v) = R + r\cos(v/r)$ and $\gamma_2(v) = r\sin(v/r)$. Since $\gamma = (\gamma_1, \gamma_2) : (0, 2\pi r) \to \mathbb{R}$ is a unit speed curve, we obtain for the geodesic equation

$$u'' = -2\frac{\gamma_1'(v)}{\gamma_1(v)}u'v' = 2\frac{\sin(v/r)}{R + r\cos(v/r)}u'v'$$

and

$$v'' = \gamma_1(v)\gamma_1'(v)u'u' = -(R + r\cos(v/r))\sin(v/r)u'u',$$

where we use the identities from Example 4.3.

Remark 4.15 We refer to M12 for techniques to solve systems of ordinary differential equations. Generally speaking, it is rather exceptional that one can explicitly write down a solution to a geodesic equation.

4.3 Curvature tensor and the Theorema Egregium

WEEK 8

In the previous section we saw that the Christoffel symbols $\Gamma^i_{jk} : U \to \mathbb{R}$, defined with respect to a local parametrisation $F : U \to \mathbb{R}^3$, encode the covariant derivative ∇. It is natural to ask what object the functions $R_{ijkl} : U \to \mathbb{R}$ encode. We first define:

Definition 4.16 (Commutator of two vector fields) The *commutator* of the two vector fields $X, Y \in \mathfrak{X}(M)$ is the vector field $[X, Y] \in \mathfrak{X}(M)$ defined by

$$[X, Y] = \nabla_X Y - \nabla_Y X. \tag{4.12}$$

Remark 4.17 The previous definition is a pedagogical simplification of the notion of the commutator of two vector fields. The commutator is usually defined in terms of the so-called flows of the vector fields. We refer to the literature for further details.

We let $C^\infty(M)$ denote the smooth functions on a surface M, that is, $f \in C^\infty(M)$ is a function $f : M \to \mathbb{R}$ which is smooth. Using the commutator we now define:

Definition 4.18 (Curvature tensor) The map $\mathcal{R} : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ defined by the rule

$$\mathcal{R} : (X, Y, Z, W) \mapsto \langle \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z, W\rangle$$

is called the *curvature tensor of* M.

The curvature tensor satisfies:

Proposition 4.19 *For a local parametrisation $F : U \to M$ we have for all $q \in U$*

$$\mathcal{R}(B_i, B_j, B_k, B_r)(F(q)) = R_{jikr}(q).$$

Proof We first want to compute $\nabla_{B_i}(\nabla_{B_j}B_k)(F(q))$ for all $q \in U$ and all $1 \leqslant i, j, k \leqslant 2$. From Proposition 4.9 we know that $(\nabla_{B_j}B_k)(F(q)) = \Gamma^l_{jk}(q)B_l(F(q))$. In order to compute $\nabla_{B_i}(\nabla_{B_j}B_k)(F(q))$ we proceed as in the proof of Proposition 4.9 and choose a curve $\gamma : (-\epsilon, \epsilon) \to M$ so that $\gamma(0) = q$ and $\dot\gamma(0) = B_i(F(q))$. We then compute

$$\frac{\mathrm{D}}{\mathrm{d}t}\Big((\nabla_{B_j}B_k) \circ \gamma\Big)(0)$$

by using (4.9). Recall that we can choose $\gamma = F \circ c$, where $c : (-\epsilon, \epsilon) \to U$ is given by $t \mapsto q + te_i$ for ϵ sufficiently small. Write $X = \big(\nabla_{B_j}B_k\big) \circ \gamma$, then $(\nabla_{B_j}B_k)(F(q)) = \Gamma^l_{jk}(q)B_l(F(q))$ implies that

$$X^l(t) = \Gamma^l_{jk}(c(t))$$

and hence

$$\frac{\mathrm{d}X^l}{\mathrm{d}t}(0) = \partial_i\Gamma^l_{jk}(q).$$

Writing (4.9) as

$$\frac{\mathrm{D}X}{\mathrm{d}t}(t) = \left(\frac{\mathrm{d}X^l}{\mathrm{d}t}(t) + \Gamma^l_{rm}(c(t))X^m(t)\frac{\mathrm{d}c^r}{\mathrm{d}t}(t)\right)\Big(\partial_l F(c(t))\Big)_{\gamma(t)},$$

we obtain

$$\frac{\mathrm{D}X}{\mathrm{d}t}(0) = \left(\partial_i\Gamma^l_{jk}(q) + \Gamma^l_{rm}(q)\Gamma^m_{jk}(q)\delta^r_i\right)\Big(\partial_l F(q)\Big)_{F(q)}$$

where we use that $\frac{\mathrm{d}c^r}{\mathrm{d}t}(0) = \delta^r_i$. In total, we get

$$\nabla_{B_i}(\nabla_{B_j}B_k)(F(q)) = \Big(\partial_i\Gamma^l_{jk}(q) + \Gamma^l_{im}(q)\Gamma^m_{jk}(q)\Big)B_l(F(q)). \tag{4.13}$$

Since $\Gamma^l_{jk} = \Gamma^l_{kj}$, it follows that $\nabla_{B_j}B_k = \nabla_{B_k}B_j$ and hence (4.12) implies $[B_j, B_k] = 0$.

Using the definition of $\mathcal{R}$ and (4.13) we thus obtain

$$\begin{aligned}\mathcal{R}(B_i, B_j, B_k, B_r)(F(q)) &= \Big\langle \nabla_{B_i}(\nabla_{B_j}B_k)(F(q)) - \nabla_{B_j}(\nabla_{B_i}B_k)(F(q)), B_r(F(q))\Big\rangle \\ &= \Big\langle\Big(\partial_i\Gamma^l_{jk}(q) + \Gamma^l_{im}(q)\Gamma^m_{jk}(q) - \partial_j\Gamma^l_{ik}(q) - \Gamma^l_{jm}(q)\Gamma^m_{ik}(q)\Big)B_l(F(q)), B_r(F(q))\Big\rangle.\end{aligned}$$

Since $\langle B_l(F(q)), B_r(F(q))\rangle = g_{lr}(q)$, this becomes

$$\begin{aligned}\mathcal{R}(B_i, B_j, B_k, B_r)(F(q)) &= \Big(\partial_i\Gamma^l_{jk}(q) + \Gamma^l_{im}(q)\Gamma^m_{jk}(q) - \partial_j\Gamma^l_{ik}(q) - \Gamma^l_{jm}(q)\Gamma^m_{ik}(q)\Big)g_{lr}(q) \\ &= R_{jikr}(q),\end{aligned}$$

where we use the definition (4.3) of the functions $R_{jikr} : U \to \mathbb{R}$. □

Remark 4.20

(i) Proposition 4.19 implies that the functions R_{ijkl} encode the curvature tensor with respect to a choice of a local parametrisation $F : U \to M$.

(ii) Notice that $\mathcal{R}$ does depend on ∇ and the first fundamental form only, it is thus an object of the intrinsic geometry of a surface.

(iii) (♡ – not examinable) Recall that second partial derivatives of a twice continuously differentiable function $f : U \to \mathbb{R}$ commute, that is, we have $\partial^2_{ij}f(q) = \partial^2_{ji}f(q)$ for all $q \in U$ and all $1 \leqslant i, j \leqslant n$. This is not true any more for second covariant derivatives. That is, in general we have $\nabla_X(\nabla_Y Z) \neq \nabla_Y(\nabla_X Z)$. The curvature tensor may be thought of as measuring the failure of second order covariant derivatives to commute. The additional term $-\nabla_{[X,Y]}Z$ in the curvature tensor makes sure that $\mathcal{R}$ is a multilinear map.

Finally, we have

Theorem 4.21 (Theorema Egregium) *Let $M \subset \mathbb{R}^3$ be a surface. The Gauss curvature of M does depend on the first fundamental form only and with respect to a choice of local parametrisation $F : U \to M$ we have for all $q \in U$*

$$K(F(q)) = \frac{R_{1212}(q)}{\det g(q)}.$$

Proof From (3.14) and (4.7) we conclude that

$$K(F(q)) = \frac{\det A(q)}{\det g(q)} = \frac{A_{11}(q)A_{22}(a) - A_{12}(q)^2}{\det g(q)} = \frac{R_{1212}(q)}{\det g(q)}.$$

We can thus express the Gauss curvature of M in terms of the curvature tensor and the first fundamental form only. Since the curvature tensor is built from ∇ and ∇ does depend on the first fundamental form only, the Gauss curvature does depend on the first fundamental form only. □

Exercise 4.22 Show that the functions R_{ijkl} satisfy the following symmetries

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}.$$

Hint: Use the Gauss equations (4.4).

From the previous exercise we conclude that

$$R_{11kl}(q) = R_{22kl}(q) = R_{ij11}(q) = R_{ij22}(q) = 0$$

for all $q \in U$ and all $1 \leqslant i, j, k, l \leqslant 2$. Theorem 4.21 implies

$$R_{1212}(q) = -R_{2112}(q) = -R_{1221}(q) = R_{2121}(q) = K(F(q)) \det g(q)$$

We thus obtain the formula

$$\boxed{R_{ijkl} = K\left(g_{ik}g_{jl} - g_{jk}g_{il}\right)}$$

which holds for all $1 \leqslant i, j, k, l \leqslant 2$ and where we omit the base point $q \in U$.

For vector fields $X, W_1, W_2 \in \mathfrak{X}(M)$ and functions $f_1, f_2 \in C^\infty(M)$, we have from the bilinearity of $\langle\cdot,\cdot\rangle_p$ for all $p \in M$ that

$$\langle X, f_1 W_1 + f_2 W_2\rangle = f_1\langle X, W_1\rangle + f_2\langle X, W_2\rangle$$

This implies that for all $X, Y, Z, W_1, W_2 \in \mathfrak{X}(M)$ and $f_1, f_2 \in C^\infty(M)$ we have

$$\mathcal{R}(X, Y, Z, f_1 W_1 + f_2 W_2) = f_1\mathcal{R}(X, Y, Z, W_1) + f_2\mathcal{R}(X, Y, Z, W_2)$$

Exercise 4.23 Show that for all $Z_1, Z_2, W \in \mathfrak{X}(M)$ and all $f_1, f_2 \in C^\infty(M)$ we have

$$\mathcal{R}(B_1, B_2, f_1 Z_1 + f_2 Z_2, W) = f_1\mathcal{R}(B_1, B_2, Z_1, W) + f_2\mathcal{R}(B_1, B_2, Z_2, W).$$

Remark 4.24 The statement from the previous exercise is still true if we replace B_1, B_2 with arbitrary vector fields X, Y, we will however not need this fact.

4.4 Geodesic curvature

As in the case of a plane curve, a closed curve $\gamma : [a, b] \to M$ in a surface M is called *simple* if the restriction of γ to the half-open interval $[a, b)$ is injective. Recall from Theorem 2.39 that a smooth unit speed curve in $\mathbb{R}^2$ that is simple and closed has rotation index ± 1 – or equivalently – total (signed) curvature $\pm 2\pi$. It is natural to ask whether this is still true for simple closed curves on a surface M. In order to turn this into a sensible question we need a notion of curvature for a curve on a surface. This leads to the notion of *geodesic curvature*.

Let $M \subset \mathbb{R}^3$ be a surface equipped with a unit normal field $N : M \to TM^{\perp}$. We define

Definition 4.25 (Geodesic curvature) Let $\gamma : I \to M$ be a smooth unit speed curve. The *geodesic curvature* of γ is the function

$$\kappa_g : I \to \mathbb{R}, \qquad t \mapsto \left\langle \frac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t}(t), N(\gamma(t)) \times \dot{\gamma}(t) \right\rangle.$$

The geodesic curvature for a curve in a surface M is indeed a natural replacement for the signed curvature of a plane unit speed curve:

Example 4.26 (Geodesic curvature of a plane curve) Let $M = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ and $\gamma = (\gamma_1, \gamma_2, 0) : I \to M$ be a unit speed curve, where $\gamma_i : I \to \mathbb{R}$ are smooth functions for $i = 1, 2$. Taking

$$N(p) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_p$$

for all $p \in M$, we obtain

$$N(\gamma(t)) \times \dot{\gamma}(t) = \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \\ 0 \end{pmatrix} \right)_{\gamma(t)} = \begin{pmatrix} -\gamma_2'(t) \\ \gamma_1'(t) \\ 0 \end{pmatrix}_{\gamma(t)}$$

and

$$\frac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t}(t) = \begin{pmatrix} \gamma_1''(t) \\ \gamma_2''(t) \\ 0 \end{pmatrix}$$

so that

$$\kappa_g(t) = -\gamma_1''(t)\gamma_2'(t) + \gamma_2''(t)\gamma_1'(t)$$

which is the signed curvature of the unit speed curve $\gamma = (\gamma_1, \gamma_2) : I \to \mathbb{R}^2$ (see (2.9)).

Exercise 4.27 Show that $\gamma : I \to M$ is a geodesic if and only if $\kappa_g(t) = 0$ for all $t \in I$.

Remark 4.28 Let $p \in M$ and $\vec{v}_p \in T_pM$. Having a unit normal field N, we let $J_p(\vec{v}_p)$ be the unique vector in T_pM so that $\vec{v}_p \times J_p(\vec{v}_p) = N(p)$. The properties of the cross product imply that this defines a linear map $J_p : T_pM \to T_pM$ which corresponds

FIGURE 4.1. A simple closed curve on a helicoid.

to "counter clockwise rotation by $\pi/2$". In particular, $J(\vec{v}_p)$ is orthogonal to $\vec{v}_p$, has the same length as $\vec{v}_p$ and we have the formula

$$N(p) \times \vec{v}_p = J(\vec{v}_p).$$

In terms of J_p, the formula for the geodesic curvature thus becomes

$$\kappa_g(t) = \left\langle \frac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t}(t), J_{\gamma(t)}(\dot{\gamma}(t)) \right\rangle,$$

Notice that this precisely corresponds to the signed curvature of a unit speed curve (2.6), where the acceleration $\ddot{\gamma}$ is replaced with the covariant derivative $\frac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t}$ of the velocity vector $\dot{\gamma}$.

For what follows it is convenient to slightly simplify notation:

Remark 4.29 (Notation)

(i) For a vector field X on M we write JX for the vector field defined by

$$JX(p) := J_p(X(p))$$

for all $p \in M$. Likewise, for a vector field Y along a curve γ we write JY for the vector field along γ defined by the rule

$$JY(t) := J_{\gamma(t)}(Y(t))$$

for all $t \in I$.

(ii) For a curve $\gamma : I \to M$ we write X_γ for the vector field along γ obtained by restricting X to $\gamma(I)$, that is,

$$X_\gamma(t) := X(\gamma(t))$$

for all $t \in I$.

Having the notion of geodesic curvature we can ask: Given a simple closed smooth unit speed curve $\gamma : [0, L] \to M$ of length L and denoting its geodesic curvature by $\kappa_g : [0, L] \to \mathbb{R}$, is it still true that

$$\int_0^L \kappa_g(t)\mathrm{d}t = \pm 2\pi?$$

To answer this question we need the notion of integrating a function over a surface M.

Let $f : M \to \mathbb{R}$ be a function and $F : U \to M$ a local parametrisation of M. Suppose $\Omega \subset U$ is a subset so that the function defined on Ω by the rule

$$h(q) := f(F(q))\sqrt{\det(g(q))}$$

is measurable in the sense of Lebesgue. Then we define

$$\int_{F(\Omega)} f\, dA := \int_{\Omega} (f \circ F)\sqrt{\det(g)}d\mu, \tag{4.14}$$

provided the right hand side is finite and where integration is carried out with respect to the Lebesgue measure.

The motivation for the factor $\sqrt{\det g}$ is a follows: Recall that $\{B_1(p), B_2(p)\}$ is a basis for all $p \in F(U)$. Consequently, $B_1(p) \times B_2(p)$ spans $T_pM^{\perp}$ for all $p \in F(U)$. Therefore we may take

$$N(p) = \left(\frac{B_1(p) \times B_2(p)}{\|B_1(p) \times B_2(p)\|}\right)_p$$

as a unit normal field on $F(U) \subset M$.

A direct calculation shows that the cross product of two column vectors $\vec{v}, \vec{w} \in M_{3,1}(\mathbb{R})$ satisfies

$$\|\vec{v} \times \vec{w}\|^2 = (\vec{v} \times \vec{w}) \cdot (\vec{v} \times \vec{w}) = (\vec{v} \cdot \vec{v})(\vec{w} \cdot \vec{w}) - (\vec{v} \cdot \vec{w})^2.$$

which implies that for all $q \in U$

$$\begin{aligned}\|B_1(p) \times B_2(p)\| &= \sqrt{\langle B_1(p), B_1(p)\rangle\langle B_2(p), B_2(p)\rangle - \langle B_1(p), B_2(p)\rangle^2}\\ &= \sqrt{(\partial_1 F(q) \cdot \partial_1 F(q))\,(\partial_2 F(q) \cdot \partial_2 F(q)) - (\partial_1 F(q) \cdot \partial_2 F(q))^2}\\ &= \sqrt{g_{11}(q)g_{22}(q) - g_{12}(q)^2} = \sqrt{\det(g(q))},\end{aligned}$$

where we write $p = F(q)$.

Recall that the quantity $\|\vec{v} \times \vec{w}\|$ equals the area of the parallelogram whose sides are given by the vectors $\vec{v}, \vec{w}$. The factor $\sqrt{\det(g(q))}$ thus gives the surface area of the parallelogram in $T_{F(q)}M$ whose sides are given by $B_1(F(q))$ and $B_2(F(q))$.

Example 4.30 (Surface area of the 2-sphere) Let $M = S^2$ be the 2-sphere of radius 1 and take $f : S^2 \to \mathbb{R}$ to be the function assuming the value 1 everywhere. For the parametrisation $F : U \to S^2 \subset \mathbb{R}^3$ from Example 3.55 with $U = (0, 2\pi) \times (-\pi/2, \pi/2)$ we computed

$$g(q) = \begin{pmatrix} \cos(v)^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

where $q = (u, v)$. Since $\cos(v) > 0$ for $v \in (-\pi/2, \pi/2)$ we thus obtain

$$\begin{aligned}\int_{F(U)} dA &= \int_U \cos(v)d\mu = \int_0^{2\pi} \left(\int_{-\pi/2}^{\pi/2} \cos(v)\mathrm{d}v\right) \mathrm{d}u\\ &= \int_0^{2\pi} \sin(v)\Big|_{-\pi/2}^{\pi/2} \mathrm{d}u = \int_0^{2\pi} 2\mathrm{d}u = 4\pi.\end{aligned}$$

If $\tilde{U} \subset \mathbb{R}^2$ is an open set and $\varphi : \tilde{U} \to U$ a diffeomorphism, then one obtains another parametrisation of M given by $\tilde{F} := F \circ \varphi : \tilde{U} \to M$. Denoting by $\tilde{\Omega}$ the subset of $\tilde{U}$ so that $\varphi(\tilde{\Omega}) = \Omega$, we have

$$\int_{\tilde{\Omega}} (f \circ \tilde{F})\sqrt{\det(\tilde{g})}d\mu = \int_{\Omega} (f \circ F)\sqrt{\det(g)}d\mu, \tag{4.15}$$

where $\tilde{g} : \tilde{U} \to M_{2,2}(\mathbb{R})$ encodes the first fundamental form with respect to $\tilde{F}$. For a proof of (4.15) we refer to a book about measure theory. A consequence of (4.15) is that the definition (4.14) is independent of the parametrisation $F : U \to M$ and we can thus define the integral of a smooth function $f : M \to \mathbb{R}$ over (sufficiently nice) subsets $D \subset M$.

4.5 First version of the Gauss–Bonnet Theorem

WEEK 9

We are now able to answer the question from the previous section.

Theorem 4.31 (First version of the Gauss–Bonnet Theorem) *Let $\gamma : [0, L] \to M$ be a smooth simple closed unit speed curve of length L whose image is contained in $F(U)$ for some local parametrisation $F : U \to M$. Let $D \subset M$ denote the region enclosed by γ and assume that $J\dot{\gamma}(t)$ points into the interior of D for all $t \in [0, L]$. Then*

$$\int_0^L \kappa_g(t)\mathrm{d}t = 2\pi - \int_D K\, dA,$$

where κ_g denotes the geodesic curvature of γ and K the Gauss curvature of M.

The proof of the Gauss–Bonnet Theorem relies on Green's theorem which we will prove in the study week 13 of this course.

Lemma 4.32 *Let $Y : [0, L] \to M$ be a vector field along the curve $\gamma : [0, L] \to M$ satisfying $\langle Y(t), Y(t)\rangle = 1$ for all $t \in [0, L]$, then we have for all $t \in [0, L]$*

$$\frac{\mathrm{D}JY}{\mathrm{d}t}(t) = J_{\gamma(t)}\left(\frac{\mathrm{D}Y}{\mathrm{d}t}(t)\right), \tag{4.16}$$

where $J_{\gamma(t)}$ is defined as in Remark 4.28.

Proof In what follows, all identities hold for all $t \in [0, L]$, we will however omit writing t each time to lighten notation. Since $1 = \langle Y, Y\rangle = \langle JY, JY\rangle$, taking the time derivative implies

$$\left\langle \frac{\mathrm{D}Y}{\mathrm{d}t}, Y\right\rangle = 0 \qquad \text{and} \qquad \left\langle \frac{\mathrm{D}JY}{\mathrm{d}t}, JY\right\rangle = 0. \tag{4.17}$$

We also have

$$\langle Y, JY\rangle = 0$$

and taking the time derivative again, this implies

$$0 = \left\langle \frac{\mathrm{D}Y}{\mathrm{d}t}, JY\right\rangle + \left\langle Y, \frac{\mathrm{D}JY}{\mathrm{d}t}\right\rangle.$$

Applying J to the left summand, we obtain

$$\left\langle Y, \frac{\mathrm{D}JY}{\mathrm{d}t}\right\rangle = -\left\langle J\left(\frac{\mathrm{D}Y}{\mathrm{d}t}\right), J(JY)\right\rangle = \left\langle Y, J\left(\frac{\mathrm{D}Y}{\mathrm{d}t}\right)\right\rangle,$$

where we use that $J(JY) = -Y$. We thus have

$$0 = \left\langle Y, \frac{\mathrm{D}JY}{\mathrm{d}t} - J\left(\frac{\mathrm{D}Y}{\mathrm{d}t}\right)\right\rangle. \tag{4.18}$$

Applying J to the first identity in (4.17) we also have

$$0 = \left\langle J\left(\frac{\mathrm{D}Y}{\mathrm{d}t}\right), JY\right\rangle$$

Using the second identity in (4.17) we conclude

$$0 = \left\langle JY, \frac{\mathrm{D}JY}{\mathrm{d}t} - J\left(\frac{\mathrm{D}Y}{\mathrm{d}t}\right)\right\rangle \tag{4.19}$$

Since $\{Y(t), JY(t)\}$ is a basis of $T_{\gamma(t)}M$ for all $t \in [0, L]$, (4.18) and (4.19) imply that the vector

$$\frac{\mathrm{D}JY}{\mathrm{d}t}(t) - J_{\gamma(t)}\left(\frac{\mathrm{D}Y}{\mathrm{d}t}(t)\right)$$

is orthogonal to all vectors of $T_{\gamma(t)}M$. Since $\langle\cdot,\cdot\rangle_{\gamma(t)}$ is non-degenerate this implies the claim. □

We also need:

Lemma 4.33 *Let $F: U \to M$ be a local parametrisation of the surface $M \subset \mathbb{R}^3$ with associated vector fields B_1, B_2 on $F(U) \subset M$, X a smooth vector field on M and $c: I \to U$ a smooth curve. Writing $\gamma = F \circ c$, we have*

$$\frac{\mathrm{D}X_\gamma}{\mathrm{d}t} = \frac{\mathrm{d}c^1}{\mathrm{d}t}(\nabla_{B_1}X)(\gamma) + \frac{\mathrm{d}c^2}{\mathrm{d}t}(\nabla_{B_2}X)(\gamma).$$

Proof Since $\{B_1(p), B_2(p)\}$ is a basis of T_pM for all $p \in F(U)$, there exist unique smooth functions $X^1, X^2: F(U) \to \mathbb{R}$ so that

$$X(p) = X^1(p)B_1(p) + X^2(p)B_2(p) = X^i(p)B_i(p),$$

for all $p \in F(U)$, where we use the summation convention on the right hand side. This gives for $j = 1, 2$

$$\nabla_{B_j}X = \nabla_{B_j}(X^iB_i) = \mathrm{d}X^i(B_j) + \nabla_{B_j}B_i = \mathrm{d}X^i(B_j)B_i + X^i\Gamma^k_{ij}B_k,$$

where we use Lemma 4.7 Item (iv) as well as Proposition 4.9 and omit writing base points. Consequently, we have

$$\begin{aligned}\frac{\mathrm{d}c^j}{\mathrm{d}t}(t)(\nabla_{B_j}X)(\gamma(t)) &= \frac{\mathrm{d}c^j}{\mathrm{d}t}(t)\left(\mathrm{d}X^i(B_j(\gamma(t)))B_i(\gamma(t)) + X^i(\gamma(t))\Gamma^k_{ij}(c(t))B_k(\gamma(t))\right)\\ &= \frac{\mathrm{D}X_\gamma}{\mathrm{d}t}(t) - B_k(\gamma(t))\left(\frac{\mathrm{d}}{\mathrm{d}t}(X^k\circ\gamma)(t)\right.\\ &\qquad \left. - \frac{\mathrm{d}c^j}{\mathrm{d}t}(t)\mathrm{d}X^k(B_j(\gamma(t))\right),\end{aligned}$$

where we use (4.9). The claim thus follows provided we show that for $k = 1, 2$ and all $t \in I$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(X^k\circ\gamma)(t) = \mathrm{d}X^k(B_j(\gamma(t)))\frac{\mathrm{d}c^j}{\mathrm{d}t}(t).$$

Since $\gamma = F \circ c$, the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(X^k\circ F\circ c\right)(t) = \begin{pmatrix}\partial_1X^k(\gamma(t)) & \partial_2X^k(\gamma(t)) & \partial_3X^k(\gamma(t))\end{pmatrix}\begin{pmatrix}\partial_1F_1(c(t)) & \partial_2F_1(c(t))\\ \partial_1F_2(c(t)) & \partial_2F_2(c(t))\\ \partial_1F_3(c(t)) & \partial_2F_3(c(t))\end{pmatrix}\begin{pmatrix}\frac{\mathrm{d}c^1}{\mathrm{d}t}(t)\\ \frac{\mathrm{d}c^2}{\mathrm{d}t}(t)\end{pmatrix} \tag{4.20}$$

which agrees with

$$\mathrm{d}X^k(B_1(\gamma(t)))\frac{\mathrm{d}c^1}{\mathrm{d}t}(t) + \mathrm{d}X^k(B_2(\gamma(t)))\frac{\mathrm{d}c^2}{\mathrm{d}t}(t) = \mathrm{d}X^k(B_j(\gamma(t)))\frac{\mathrm{d}c^j}{\mathrm{d}t}(t),$$

since $B_j(\gamma(t)) = \left(\partial_jF(c(t))\right)_{\gamma(t)}$. □

Proof of Theorem 4.31 Let Z be the vector field defined on $F(U)$ by the rule

$$Z(F(q)) = \frac{\partial_1F(q)}{\sqrt{g_{11}(q)}}_{F(q)}$$

for all $q \in U$. Notice that $\langle Z(F(q)), Z(F(q))\rangle = 1$ and

$$Z(F(q)) = \frac{B_1(F(q))}{\sqrt{g_{11}(q)}}$$

for all $q \in U$.

Since $\gamma : [0, L] \to M$ is a unit speed curve, there exists a polar angle function $\phi : [0, L] \to \mathbb{R}$ so that

$$\dot{\gamma} = \cos(\phi) Z_\gamma + \sin(\phi) J Z_\gamma \tag{4.21}$$

where here and henceforth we omit writing the time $t \in [0, L]$. From this we compute

$$\frac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t} = \sin(\phi)\left(\frac{\mathrm{D}JZ_\gamma}{\mathrm{d}t} - \phi' Z_\gamma\right) + \cos(\phi)\left(\frac{\mathrm{D}Z_\gamma}{\mathrm{d}t} + \phi' JZ_\gamma\right).$$

Using this identity and Lemma 4.32 together with

$$J\dot{\gamma} = \cos(\phi) JZ_\gamma - \sin(\phi) Z_\gamma,$$

We can calculate that

$$\kappa_g = \left\langle \frac{\mathrm{D}\dot{\gamma}}{\mathrm{d}t}, J\dot{\gamma} \right\rangle = \phi' + \left\langle \frac{\mathrm{D}Z_\gamma}{\mathrm{d}t}, JZ_\gamma \right\rangle.$$

We next want to evaluate $\frac{\mathrm{D}Z_\gamma}{\mathrm{d}t}$ using Lemma 4.33. For this we need expressions for $\nabla_{B_i} Z$ for $i = 1, 2$. We obtain

$$0 = \mathrm{d}\left(\langle Z, Z \rangle\right) = 2\langle \nabla_{B_i} Z, Z \rangle,$$

where we use (4.11). It follows that $(\nabla_{B_i} Z)(p)$ is orthogonal to $Z(p)$ for all $p \in F(U)$ and hence there exist unique functions $P : U \to \mathbb{R}$ and $Q : U \to \mathbb{R}$ so that

$$(\nabla_{B_1} Z)(F(q)) = P(q) JZ(F(q)) \qquad \text{and} \qquad (\nabla_{B_2} Z)(F(q)) = Q(q) JZ(F(q))$$

Let $c : [0, L] \to U$ be the smooth curve so that $\gamma = F \circ c$. Using Lemma 4.33 we thus obtain

$$\frac{\mathrm{D}Z_\gamma}{\mathrm{d}t}(t) = \frac{\mathrm{d}c^1}{\mathrm{d}t}(t) P(c(t)) JZ(\gamma(t)) + \frac{\mathrm{d}c^2}{\mathrm{d}t} Q(c(t)) JZ(\gamma(t))$$

hence we have

$$\left\langle \frac{\mathrm{D}Z_\gamma}{\mathrm{d}t}(t), JZ_\gamma(t) \right\rangle = \frac{\mathrm{d}c^1}{\mathrm{d}t}(t) P(c(t)) + \frac{\mathrm{d}c^2}{\mathrm{d}t}(t) Q(c(t)).$$

Now Green's theorem states that

$$\int_0^L \left(\frac{\mathrm{d}c^1}{\mathrm{d}t}(t) P(c(t)) + \frac{\mathrm{d}c^2}{\mathrm{d}t}(t) Q(c(t)) \right) \mathrm{d}t = \int_D \partial_1 Q(q) - \partial_2 P(q) d\mu.$$

Using the expressions for $\nabla_{B_i} Z$ we compute

$$\begin{aligned} \left(\nabla_{B_1}(\nabla_{B_2} Z)\right)(F(q)) &= \partial_1 Q(q) JZ(F(q)) + Q(q) J\nabla_{B_1} Z(F(q)), \\ &= \partial_1 Q(q) JZ(F(q)) - P(q) Q(q) Z(F(q)) \end{aligned}$$

and

$$\begin{aligned} \left(\nabla_{B_2}(\nabla_{B_1} Z)\right)(F(q)) &= \partial_2 P(q) JZ(F(q)) + P(q) J\nabla_{B_2} Z(F(q)), \\ &= \partial_2 P(q) JZ(F(q)) - P(q) Q(q) Z(F(q)) \end{aligned}$$

so that

$$\begin{aligned} \partial_1 Q(q) - \partial_2 P(q) &= \langle \nabla_{B_1}(\nabla_{B_2} Z) - \nabla_{B_1}(\nabla_{B_2} Z), JZ \rangle (F(q)) \\ &= \mathcal{R}(B_1, B_2, Z, JZ)(F(q)), \end{aligned}$$

where we use that $[B_1, B_2] = 0$, since the Christoffel symbols satisfy $\Gamma^i_{jk} = \Gamma^i_{kj}$.

It remains to compute $\mathcal{R}(B_1, B_2, Z, JZ)$. Recall that

$$Z = \frac{B_1}{\sqrt{g_{11}}}.$$

From the conditions $\langle Z, JZ \rangle = 0$, $\langle JZ, JZ \rangle = 1$ and $Z \times JZ = N$ we obtain with a calculation that we must have

$$JZ = \frac{1}{\sqrt{\det(g)}\sqrt{g_{11}}} \left(g_{11} B_2 - g_{12} B_1 \right).$$

Using these expressions, we obtain

$$\begin{aligned}\mathcal{R}(B_1, B_2, Z, JZ) &= \frac{1}{\sqrt{\det(g)}g_{11}}\mathcal{R}(B_1, B_2, B_1, g_{11}B_2 - g_{12}B_1) \\ &= \frac{1}{\sqrt{\det(g)}}\mathcal{R}(B_1, B_2, B_1, B_2) = \frac{R_{2112}}{\sqrt{\det(g)}} = -\frac{R_{1212}}{\sqrt{\det(g)}} \\ &= -K\sqrt{\det g}\end{aligned}$$

where we use Proposition 4.19, Theorem 4.21 and Exercise 4.22. In summary, we have calculated that

$$\int_0^L \kappa_g(t)\mathrm{d}t = \int_0^L \phi'(t)\mathrm{d}t - \int_D KdA.$$

Since $J\dot{\gamma}(t)$ points into the interior of D for all $t \in [0, L]$, it follows with Theorem 2.39 that $\int_0^L \phi'(t)\mathrm{d}t = 2\pi$. □

4.6 Second version of the Gauss–Bonnet Theorem

Recall that one of the fundamental theorems of elementary geometry states that the sum of the interior angles of a triangle equals π. A triangle consists of three distinct points (often called vertices) in the plane $\mathbb{R}^2$ which are connected by segments of straight lines (often called edges). In the context of a surface $M \subset \mathbb{R}^3$, the notion of a straight line is replaced by the notion of a geodesic. This leads to the notion of a *geodesic triangle*.

Definition 4.34 (Geodesic triangle) A *geodesic triangle* $\partial\Delta$ on an oriented surface $M \subset \mathbb{R}^3$ consists of three distinct points $p_1, p_2, p_3 \in M$ connected by segments of geodesics. That is, there exist geodesics $\gamma_i : [0, \ell_i] \to M$ with $\gamma_i(0) = p_i$ and $\gamma_i(\ell_i) = p_{i+1}$ (with the convention that $p_4 = p_1$). Furthermore, $\gamma_i : [0, \ell_i] \to M$ is assumed to be injective.

We define the *exterior angle* at p_i to be the angle between the vectors $\dot{\gamma}_{i-1}(\ell_{i-1})$ and $\dot{\gamma}_i(0)$ with the convention $\gamma_0 = \gamma_3$ and $\ell_0 = \ell_3$. The exterior angle is negative when $\dot{\gamma}_{i-1}(\ell_{i-1}) \times \dot{\gamma}_i(0)$ is a negative multiple of $N(p_i)$. Here and henceforth we always assume that $-\pi < \vartheta_i < \pi$. The *interior angle* α_i at p_i is then defined to be $\alpha_i = \pi - \vartheta_i$.

Example 4.35 (Geodesic triangle on the sphere) On $S^2 \subset \mathbb{R}^3$ we consider a *octant*, that is, the region enclosed by a geodesic triangle with $p_1 = (1, 0, 0)$, $p_2 = (0, 1, 0)$ and $p_3 = (0, 0, 1)$. Here we may take geodesics

$$\begin{aligned}\gamma_1(t) &: [0, \pi/2] \to S^2, \qquad t \mapsto \cos(t)p_1 + \sin(t)p_2, \\ \gamma_2(t) &: [0, \pi/2] \to S^2, \qquad t \mapsto \cos(t)p_2 + \sin(t)p_3, \\ \gamma_3(t) &: [0, \pi/2] \to S^2, \qquad t \mapsto \cos(t)p_3 + \sin(t)p_1.\end{aligned}$$

It follows with a simple calculation that $\alpha_1 = \alpha_2 = \alpha_3 = \pi/2$ so that

$$\alpha_1 + \alpha_2 + \alpha_3 = \frac{3\pi}{2} > \pi.$$

For a geodesic triangle $\partial\Delta$ it is thus not true anymore that the sum of interior angles is always π. It is natural to guess that the angle deficit between π and the sum of interior angles is related to the curvature of the enclosed region Δ. This suggests to look into a version of the Gauss–Bonnet Theorem for curves γ that are only *piecewise smooth*.

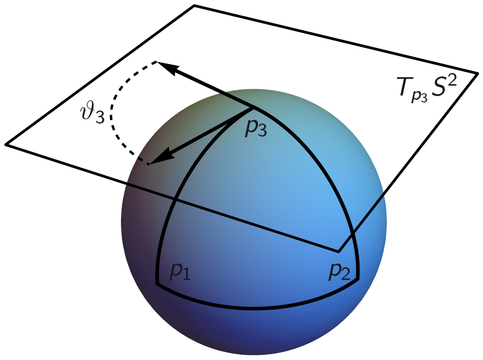

FIGURE 4.2. A geodesic triangle on the 2-sphere enclosing an octant and its exterior angle ϑ_3.

Roughly speaking, these are curves that are smooth except for finitely many exception points, called *corners*.

Definition 4.36 (Piecewise smooth curve) A curve $\gamma : [a, b] \to M$ is called *piecewise smooth* if there exists $k \in \mathbb{N}$ and times $a = T_0 < T_1 < \cdots < T_k = b$ so that $\gamma|_{[T_i, T_{i+1}]} : [T_i, T_{i+1}] \to M$ is smooth.

Notice that if $\gamma : [a, b] \to M$ is a geodesic and $\varphi : \mathbb{R} \to \mathbb{R}$ a smooth parameter of the form $\varphi(t) = st + t_0$ for real numbers s, t_0, then $\gamma \circ \varphi$ is also a geodesic.

We define the exterior angle at the corner of a piecewise smooth curve as in the case of a geodesic triangle.

Example 4.37 A geodesic triangle may be thought of as a piecewise smooth curve.

We now have:

Theorem 4.38 (Second version of the Gauss–Bonnet Theorem) *Let $\gamma : [0, L] \to M$ be a simple closed unit speed curve of length L which is piecewise smooth with exterior angles $\vartheta_1, \dots, \vartheta_k$ at the corners $p_1, \dots, p_k$ of γ and whose image is contained in $F(U)$ for some local parametrisation $F : U \to M$. Let D denote the region enclosed by γ and assume that $J\dot{\gamma}(t)$ points into the interior of D for all $t \in [0, L]$ with the exception of the corner points. Then*

$$\int_0^L k_g(t)\mathrm{d}t + \sum_{i=1}^{k} \vartheta_i = 2\pi - \int_D K dA,$$

where k_g denotes the geodesic curvature of γ and K the Gauss curvature of M.

This version of the Gauss–Bonnet Theorem implies:

Corollary 4.39 *Let $\partial\Delta \subset F(U)$ be a geodesic triangle enclosing the region $D \subset M$ and let α_i denote the interior angle at the corner p_i of $\partial\Delta$, where $i = 1, 2, 3$. Then*

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi + \int_D K dA.$$

Proof Since $\partial\Delta$ is geodesic triangle, the geodesic curvature terms in Theorem 4.38 are all zero, hence we obtain

$$\vartheta_1 + \vartheta_2 + \vartheta_3 = 2\pi - \int_D K dA.$$

Since the exterior angle α_i satisfies $\alpha_i = \pi - \vartheta_i$ we have equivalently

$$3\pi - \alpha_1 - \alpha_2 - \alpha_3 = 2\pi - \int_D K dA \quad \Longleftrightarrow \quad \alpha_1 + \alpha_2 + \alpha_3 = \pi + \int_D K dA.$$

□

Sketch of a proof of Theorem 4.38 The curve γ is the image of a piecewise smooth curve $c : [0, L] \to U$. Let $q_1, \ldots, q_k \in U$ denote the corners of c. We can *smoothen* the curve γ as follows. For $\epsilon > 0$ sufficiently small we remove the part of c whose image is contained in a disk of radius ϵ around q_i and *glue in a smooth curve piece* to obtain a smooth curve $c_\epsilon : [0, L] \to U$ with corresponding smooth image curve $\gamma_\epsilon = F \circ c_\epsilon$. We then apply Theorem 4.31 to γ_ϵ and consider the limit as ϵ goes to zero. Let $\phi_\epsilon : [0, L] \to \mathbb{R}$ denote the polar angle function of γ_ϵ as defined by (4.21) and $k_{g,\epsilon} : [0, L] \to \mathbb{R}$ the geodesic curvature of γ_ϵ. Applying Theorem 4.31 we have

$$\int_0^L \kappa_{g,\epsilon}(t) \mathrm{d}t = \int_0^L \phi'_\epsilon(t) \mathrm{d}t - \int_{D_\epsilon} K dA, \tag{4.22}$$

where D_ϵ denote the region enclosed by γ_ϵ. As ϵ tends to zero the polar angle function ϕ_ϵ converges to a function which jumps by the exterior angle ϑ_i at each corner p_i and hence misses the contribution of ϑ_i at each corner p_i. Taking the limit as ϵ goes to 0 in (4.22) we thus arrive at

$$\int_0^L \kappa_g(t) \mathrm{d}t = 2\pi - \sum_{i=1}^k \vartheta_i - \int_D K dA.$$

□

Example 4.40 (Example 4.35 continued) For the 2-sphere S^2 of radius 1 we have $K = 1$ and hence for the octant Δ from Example 4.35 we have

$$\alpha_1 + \alpha_2 + \alpha_3 = \frac{3\pi}{2} = \pi + \int_\Delta K dA = \pi + \int_\Delta dA,$$

so that $\int_\Delta dA = \pi/2$, that is, the octant has surface area $\pi/2$ and hence the whole sphere has surface area $8 \cdot \pi/2 = 4\pi$, which is in agreement with the calculation in Example 4.30.

Exercise 4.41 Use the Gauss–Bonnet theorem to conclude that on a surface M with $K(p) < 0$ two geodesics $\gamma_1 : [0, L_1] \to M$ and $\gamma_2 : [0, L_2] \to M$ can intersect in at most one point.

4.7 Global version of the Gauss–Bonnet Theorem

WEEK 10

It is natural to wonder whether the Gauss–Bonnet Theorem considered so far has any implications on the *total Gauss curvature* of an embedded surface $M \subset \mathbb{R}^3$. This is indeed the case. In case M is *compact* (recall this this is equivalent to $M \subset \mathbb{R}^3$ being a closed and bounded subset), the total Gauss curvature is always an integer multiple of 2π. The integer is called the *Euler characteristic of M*. It can be computed in terms of a so-called *triangulation of M*. A subset Δ of a surface M is called a *triangle* if Δ is the image of a simple closed curve which is a piecewise smooth curve and which has 3 corners. A *triangulation of a surface M* is a finite set $\mathcal{T} = \{\Delta_i \subset M \,|\, 1 \leqslant i \leqslant N\}$ of triangles on M so that

(i) $\bigcup_{i=1}^N \Delta_i = M$;
(ii) if for a pair of indices $i \neq j$ we have $\Delta_i \cap \Delta_j \neq \emptyset$, then $\Delta_i \cap \Delta_j$ consists of a common edge or of a common vertex.

For a given triangulation $\mathcal{T}$ of M we call

$$\chi = V - E + F$$

the *Euler characteristic* of the triangulation $\mathcal{T}$. Here $F = N$ denotes the number of faces (i.e. triangles) of $\mathcal{T}$. The number V denotes the number of vertices and E the number of edges of $\mathcal{T}$.

An important theorem from a course about topology states that every compact embedded surface $M \subset \mathbb{R}^3$ admits a triangulation $\mathcal{T}$ and moreover $\chi = \chi(M)$ is independent of $\mathcal{T}$. Furthermore, one can show that $\chi(M)$ is related to the (roughly speaking) number g of *holes* of the surface via the relation

$$\chi(M) = 2 - 2g.$$

Example 4.42 For the 2-sphere $S^2 \subset \mathbb{R}^3$ we obtain a triangulation $\mathcal{T}$ in terms of its octants and for this triangulation we have

$$\chi(S^2) = V - E + F = 6 - 12 + 8 = 2.$$

A sphere has no hole, hence $\chi(M) = 2 - 2 \cdot 0 = 2$, which is in agreement with the value obtained in terms of a triangulation.

Example 4.43 The torus $T \subset \mathbb{R}^3$ has 1 hole, hence

$$\chi(T) = 2 - 2 \cdot 1 = 0.$$

Considering surfaces with more than one hole we obtain surfaces whose Euler characteristic is negative.

We can now state:

Theorem 4.44 (Global version of the Gauss–Bonnet theorem) *Let $M \subset \mathbb{R}^3$ be a compact embedded surface with Gauss curvature K, then*

$$\frac{1}{2\pi}\int_M K dA = \chi(M).$$

Sketch of a proof Since M is compact we can find a triangulation $\mathcal{T}$ of M so that each triangle $\Delta_i \in \mathcal{T}$ is contained in the image of a local parametrisation of M. Applying the local version of the Gauss–Bonnet Theorem 4.38, we obtain

$$\int_{\Delta_i} K dA + \int_{\partial \Delta_i} \kappa_g(t) \mathrm{d}t = 2\pi - \sum_{j=1}^{3} \vartheta_{ij},$$

where here $\int_{\partial \Delta_i} \kappa_g(t)\mathrm{d}t$ stands for the first summand in Theorem 4.38 with $\gamma : [0, L] \to M$ being a simple closed unit speed curve travelling counter clockwise around the triangle Δ_i. Moreover ϑ_{ij} denotes the j-th exterior angle of the i-th triangle Δ_i. Having F triangles in our triangulation, we thus obtain

$$\sum_{i=1}^{F} \left(\int_{\Delta_i} K dA + \int_{\partial \Delta_i} \kappa_g(t) \mathrm{d}t \right) = 2\pi F - \sum_{i=1}^{F} \sum_{j=1}^{3} \vartheta_{ij}.$$

In the second summand, the geodesic curvature is integrated over each edge twice, with opposing orientation. Consequently

$$0 = \sum_{i=1}^{F} \int_{\partial \Delta_i} \kappa_g(t) \mathrm{d}t$$

and we obtain

$$\sum_{i=1}^{F} \int_{\Delta_i} K dA = \int_M K dA = 2\pi F - \sum_{i,j} \vartheta_{ij} = 2\pi F - \sum_{i,j} (\pi - \alpha_{ij}),$$

where α_{ij} denotes the j-th interior angle of the i-th triangle Δ_i and where we write $\sum_{i,j}$ instead of $\sum_{i=1}^{F} \sum_{j=1}^{3}$ Notice that the sum of all interior angles at each vertex of $\mathcal{T}$ is 2π. This implies that $\sum_{i,j} \alpha_{ij} = 2\pi V$, where V denotes the number of vertices of the triangulation $\mathcal{T}$. We thus arrive at

$$\int_M K dA = 2\pi \left(F + V - \frac{3}{2} F \right),$$

where we use that $\sum_{i,j} \pi = 3F\pi$. Since every edge of the triangulation belongs two exactly two triangles and a triangle has 3 edges, we must have

$$3F = 2E$$

so that

$$\frac{1}{2\pi} \int_K dA = V - E + F = \chi(M),$$

as claimed. □

CHAPTER 5

Further topics

In this chapter we provide an outlook to some further topics in differential geometry which are typically studied in depth in a master course. *The content of this chapter is not examinable*.

5.1 Differential forms

The proof of Gauss' Theorema Egregium can be simplified by using so-called *differential forms*. We start with a brief introduction to differential forms.

Recall that a vector field associates to every point p of its domain of definition a tangent vector $X(p)$ in the corresponding tangent space. Closely related is the notion of a 1-form:

Definition 5.1 (1-form) Let $\mathcal{X} \subset \mathbb{R}^n$ be a subset. A 1-form α on $\mathcal{X}$ is a map $\alpha : \mathcal{X} \to T^*\mathbb{R}^n$ so that $\alpha|_p := \alpha(p) \in T_p^*\mathbb{R}^n$ for all $p \in U$. Writing

$$\alpha|_p = \alpha_1(p)\mathrm{d}x_1|_p + \cdots + \alpha_n(p)\mathrm{d}x_n|_p$$

for functions $\alpha_i : \mathcal{X} \to \mathbb{R}$, where $1 \leqslant i \leqslant n$. We call α smooth if the functions α_i are smooth for all $1 \leqslant i \leqslant n$.

Example 5.2 (Exterior derivative) Let $f : \mathcal{X} \to \mathbb{R}$ be a smooth function, then its exterior derivative $\mathrm{d}f$ is a smooth 1-form on $\mathcal{X}$.

1-forms can be added and multiplied with functions in the obvious way. If α, β are 1-forms on $\mathcal{X}$ and $f : \mathcal{X} \to \mathbb{R}$ a function, then we define

$$(\alpha + \beta)(\vec{v}_p) = \alpha(\vec{v}_p) + \beta(\vec{v}_p),$$
$$(f\alpha)(\vec{v}_p) = (\alpha f)(\vec{v}_p) = f(p)\alpha(\vec{v}_p)$$

for all $p \in \mathcal{X}$ and $\vec{v}_p \in T_p\mathbb{R}^n$.

Given 1-forms $\alpha, \beta : \mathcal{X} \to T^*\mathbb{R}^n$, we can define a symmetric and alternating bilinear form on each tangent space $T_p\mathbb{R}^n$ for $p \in \mathcal{X}$. For all $p \in \mathcal{X}$ and $\vec{v}_p, \vec{w}_p \in T_p\mathbb{R}^n$, we define $(\alpha\beta)|_p : T_p\mathbb{R}^n \times T_p\mathbb{R}^n \to \mathbb{R}$ by the rule

$$(\alpha\beta)|_p(\vec{v}_p, \vec{w}_p) = \frac{1}{2}\left(\alpha(\vec{v}_p)\beta(\vec{w}_p) + \beta(\vec{w}_p)\alpha(\vec{v}_p)\right).$$

Notice that for all $p \in \mathcal{X}$ the map $(\alpha\beta)|_p$ is a symmetric bilinear form on $T_p\mathbb{R}^n$. Similarly we can define an alternating bilinear form $(\alpha \wedge \beta)_p : T_p\mathbb{R}^n \times T_p\mathbb{R}^n \to \mathbb{R}$ by the rule

$$(\alpha \wedge \beta)|_p(\vec{v}_p, \vec{w}_p) = \alpha(\vec{v}_p)\beta(\vec{w}_p) - \beta(\vec{v}_p)\alpha(\vec{w}_p).$$

Definition 5.3 (Wedge product) We call $\alpha \wedge \beta$ the *wedge product* of the two 1-forms α, β.

Exercise 5.4 Let α, β, ξ be 1-forms on $\mathcal{X}$ and $f, h : \mathcal{X} \to \mathbb{R}$ functions. Show that

(i) $\alpha \wedge \beta = -\beta \wedge \alpha$ so that $\alpha \wedge \alpha = 0$;

(ii) $(\alpha + \beta) \wedge \xi = \alpha \wedge \xi + \beta \wedge \xi$;

(iii) $(f\alpha) \wedge \beta = \alpha \wedge (f\beta) = f(\alpha \wedge \beta)$.

For an $\mathbb{R}$-vector space V we write $\Lambda^2(V^*)$ for the (vector space of) alternating bilinear forms on V and

$$\Lambda^2(T^*\mathbb{R}^n) := \bigcup_{p \in \mathbb{R}^n} \Lambda^2(T_p^*\mathbb{R}^n)$$

Likewise we write $S^2(V^*)$ for the symmetric bilinear forms on V and

$$S^2(T^*\mathbb{R}^n) := \bigcup_{p \in \mathbb{R}^n} S^2(T_p^*\mathbb{R}^n)$$

Definition 5.5 (2-form) Let $\mathcal{X} \subset \mathbb{R}^n$ be a subset. A 2-form on $\mathcal{X}$ is a map

$$\xi : \mathcal{X} \to \Lambda^2(T^*\mathbb{R}^n)$$

so that $\xi|_p := \xi(p) \in \Lambda^2(T_p^*\mathbb{R}^n)$.

A 2-form thus assigns to each point $p \in \mathcal{X}$ an alternating bilinear map on $T_p\mathbb{R}^n$. The wedge product $\alpha \wedge \beta$ of two 1-forms α, β is a 2-form. Moreover, we can turn every smooth 1-form into a 2-form by taking the exterior derivative:

Definition 5.6 (Exterior derivative for 1-forms) Let α be a smooth 1-form on $\mathcal{X} \subset \mathbb{R}^n$ so that $\alpha = \sum_{i=1}^n \alpha_i \mathrm{d}x_i$ for smooth functions $\alpha_i : \mathcal{X} \to \mathbb{R}$, where $1 \leqslant i \leqslant n$. The exterior derivative $\mathrm{d}\alpha$ of α is the 2-form defined as

$$\mathrm{d}\alpha|_p = \sum_{j=1}^n \sum_{i=1}^n \frac{\partial \alpha_i}{\partial x_j}(p)(\mathrm{d}x_j \wedge \mathrm{d}x_i)|_p.$$

Recall that second derivatives of a twice continuously differentiable function commute and this has the important consequence that $\mathrm{d}^2 = 0$, that is:

Lemma 5.7 *Let $f : \mathcal{X} \to \mathbb{R}$ be a smooth function. Then*

$$\mathrm{d}^2 f := \mathrm{d}(\mathrm{d}f) = 0.$$

Proof By definition we have

$$\mathrm{d}f|_p = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)\mathrm{d}x_i|_p$$

and hence

$$\mathrm{d}(\mathrm{d}f)|_p = \sum_{j=1}^n \sum_{i=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i}(p)(\mathrm{d}x_i \wedge \mathrm{d}x_j)|_p.$$

Since f is twice continuously differentiable, it follows that

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(p) = \frac{\partial^2 f}{\partial x_i \partial x_j}(p),$$

but since $(\mathrm{d}x_i \wedge \mathrm{d}x_j)|_p = -(\mathrm{d}x_j \wedge \mathrm{d}x_i)|_p$ we conclude that $\mathrm{d}(\mathrm{d}f) = 0$. □

We also have:

Lemma 5.8 (Product rule for the exterior derivative) *For a smooth* 1*-form* α *on* $\mathcal{X}$ *and a smooth function* $f : \mathcal{X} \to \mathbb{R}$ *we have*

$$\mathrm{d}(f\alpha) = \mathrm{d}f \wedge \alpha + f\mathrm{d}\alpha.$$

Proof Writing $\alpha = \alpha_i \mathrm{d}x_i$ for smooth functions $\alpha_i : \mathcal{X} \to \mathbb{R}$, we have

$$\begin{aligned}
\mathrm{d}(f\alpha) = \mathrm{d}\left(\sum_{i=1}^n f\alpha_i \mathrm{d}x_i\right) &= \sum_{j=1}^n \sum_{i=1}^n \frac{\partial(f\alpha_i)}{\partial x_j}\mathrm{d}x_j \wedge \mathrm{d}x_i \\
&= \sum_{j=1}^n \sum_{i=1}^n \left(\frac{\partial f}{\partial x_j}\alpha_i + f\frac{\partial \alpha_i}{\partial x_j}\right)\mathrm{d}x_j \wedge \mathrm{d}x_i \\
&= \left(\sum_{j=1}^n \frac{\partial f}{\partial x_j}\mathrm{d}x_j\right) \wedge \left(\sum_{i=1}^n \alpha_i \mathrm{d}x_i\right) + f\sum_{j=1}^n \sum_{i=1}^n \frac{\partial \alpha_i}{\partial x_j}\mathrm{d}x_j \wedge \mathrm{d}x_i \\
&= \mathrm{d}f \wedge \alpha + f\mathrm{d}\alpha,
\end{aligned}$$

as claimed. □

Remark 5.9 (Notation for vector-valued maps and forms)

(i) Whenever we have a smooth map $f : \mathcal{X} \to M_{n,1}(\mathbb{R})$ we write $\mathrm{d}f$ for the 1-form with values in $M_{n,1}(\mathbb{R})$ defined by the rule

$$\mathrm{d}f(\vec{v}_p) = \begin{pmatrix} \mathrm{d}f_1(\vec{v}_p) \\ \vdots \\ \mathrm{d}f_n(\vec{v}_p) \end{pmatrix} \quad \text{where} \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

for smooth functions $f_i : \mathcal{X} \to \mathbb{R}, 1 \leqslant i \leqslant n$ and where $\vec{v}_p$ in $T_p\mathbb{R}^n$ for $p \in \mathcal{X}$.

(ii) If α is a vector-valued 1-form on $\mathcal{X}$ so that

$$\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

for 1-forms α_i on $\mathcal{X}$, $1 \leqslant i \leqslant n$, then we write

$$\alpha \cdot f = f \cdot \alpha = f_1\alpha_1 + \cdots + f_n\alpha_n.$$

(iii) If β is a 1-form on $\mathcal{X}$, then we write

$$\mathrm{d}f \wedge \beta = \begin{pmatrix} \mathrm{d}f_1 \wedge \beta \\ \vdots \\ \mathrm{d}f_n \wedge \beta \end{pmatrix}$$

5.2 The Theorema Egregium revisited

We will next give a proof of the Theorema Egregium using differential forms.

Recall that $\{(\vec{e}_1)_p, (\vec{e}_2)_p, (\vec{e}_3)_p\}$ denotes the standard basis on $T_p\mathbb{R}^3$ for each $p \in \mathbb{R}^3$. In the presence of a surface $M \subset \mathbb{R}^3$ it is useful to modify this basis so that for $p \in M$ the vectors $\{(\vec{e}_1)_p, (\vec{e}_2)_p\}$ are an orthonormal basis of T_pM and so that $(\vec{e}_3)_p$ spans the normal space $T_pM^{\perp}$. We will call such a basis *adapted to* T_pM. Unfortunately it is not always possible to find an adapted basis for each tangent space of M which varies continuously over the whole surface. A theorem which goes beyond the content of this course – sometimes called the *Hairy ball theorem* – states that on the 2-sphere every continuous vector field must attain the zero tangent vector at some point. This implies in particular that we cannot find a basis $\{(\vec{e}_1)_p, (\vec{e}_2)_p\}$ for each tangent space of S^2 which varies continuously over all of S^2. We do however obtain an adapted basis locally. To see this we choose a local parametrisation $F : U \to M \subset \mathbb{R}^3$ and compute $F_u, F_v : U \to M_{3,1}(\mathbb{R})$. Recall that $\{(F_u)_{F(q)}, (F_u)_{F(q)}\}$ is basis of $T_{F(q)}M$ for all $q \in U$. Applying the Gram-Schmidt orthonormalisation procedure we thus obtain an orthonormal basis on each tangent space T_pM where $p \in F(U)$. Taking the cross product of the two tangent vectors we obtain an adapted basis for each tangent space in $F(U)$. If we forget about the base points we obtain three column vector-valued maps

$$\vec{e}_i : M \to M_{3,1}(\mathbb{R}), \qquad i = 1, 2, 3$$

where here – for notational simplicity – we pretend that these maps are defined on all of M.

Recall the map $\Psi_n : \mathbb{R}^n \to M_{n,1}(\mathbb{R})$ which turns a point into a column vector

$$\Psi_n : \mathbb{R}^n \to M_{n,1}(\mathbb{R}), \quad (x_1, \dots, x_n) \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

For what follows we consider the case $n = 3$ write $\Psi_3 : \mathbb{R}^3 \to M_{3,1}(\mathbb{R})$ as

$$\Psi_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where the functions $x, y, z : \mathbb{R}^3 \to \mathbb{R}$ denote the projection onto the respective component. Observe that

$$[\mathrm{d}\Psi_3(\vec{v}_p)]_p = \begin{pmatrix} \mathrm{d}x(\vec{v}_p) \\ \mathrm{d}y(\vec{v}_p) \\ \mathrm{d}z(\vec{v}_p) \end{pmatrix}_p = \vec{v}_p \tag{5.1}$$

for all $\vec{v}_p \in T\mathbb{R}^3$. This implies that for each $\vec{v}_p \in T_pM$ we have $[\mathrm{d}\Phi(\vec{v}_p)]_p \in T_pM$, hence there are unique smooth 1-forms ω_1, ω_2 on M so that

$$\mathrm{d}\Psi_3(\vec{v}_p) = \vec{e}_1(p)\omega_1(\vec{v}_p) + \vec{e}_2(p)\omega_2(\vec{v}_p) \tag{5.2}$$

for all $p \in M$ and all $\vec{v}_p \in T_pM$. Notice that for all $p \in M$ and all $\vec{v}_p \in T_pM$ we have

$$\omega_1(\vec{v}_p) = \langle \vec{v}_p, (\vec{e}_1)_p \rangle_p \qquad \text{and} \qquad \omega_2(\vec{v}_p) = \langle \vec{v}_p, (\vec{e}_2)_p \rangle_p \tag{5.3}$$

so that the 1-forms ω_1, ω_2 are *intrinsic quantities*. Notice the identities

$$\omega_1 = \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} \cdot \vec{e}_1 \qquad \text{and} \qquad \omega_2 = \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} \cdot \vec{e}_2. \tag{5.4}$$

which follow from computing the inner product of (5.2) with $\vec{e}_1$,$\vec{e}_2$, respectively and using (5.1).

Example 5.10 Recall that for the hyperbolic paraboloid at $p = (x, y, xy) \in M$ an orthonormal basis of T_pM is given by

$$(\vec{e}_1)_p = \frac{1}{\sqrt{1+y^2}} \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix}_p \qquad \text{and} \qquad (\vec{e}_2)_p = \frac{1}{\sqrt{1+x^2+y^2}} \begin{pmatrix} -xy/\sqrt{1+y^2} \\ \sqrt{1+y^2} \\ x/\sqrt{1+y^2} \end{pmatrix}_p .$$

Using (5.4) we compute that

$$\omega_1 = \frac{1}{\sqrt{1+y^2}}(\mathrm{d}x + y\mathrm{d}z)$$

and

$$\omega_2 = \frac{1}{\sqrt{1+x^2+y^2}} \left(-\frac{xy}{\sqrt{1+y^2}}\mathrm{d}x + \sqrt{1+y^2}\mathrm{d}y + \frac{x}{\sqrt{1+y^2}}\mathrm{d}z \right)$$

Likewise, there exist unique 1-forms ω_{ij} on M for $1 \leqslant i, j \leqslant 3$ so that

$$\mathrm{d}\vec{e}_i(\vec{v}_p) = \sum_{k=1}^{3} \vec{e}_k(p)\omega_{ki}(\vec{v}_p)$$

for all $p \in M$ and $\vec{v}_p \in T_pM$. Omitting the tangent vector $\vec{v}_p$ we have

$$\mathrm{d}\vec{e}_i = \sum_{k=1}^{3} \vec{e}_k\omega_{ki}. \tag{5.5}$$

Since $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$ we obtain

$$0 = \mathrm{d}\left(\vec{e}_i \cdot \vec{e}_j\right) = \left(\sum_{k=1}^{3} \vec{e}_k\omega_{ki}\right) \cdot \vec{e}_j + \vec{e}_i \cdot \left(\sum_{k=1}^{3} \vec{e}_k\omega_{kj}\right) = \omega_{ji} + \omega_{ij}$$

so that

$$\omega_{ij} = -\omega_{ji}.$$

We also have:

Lemma 5.11 *At each point $p \in M$ the two cotangent vectors $\omega_1|_p, \omega_2|_p \in T_p^*M$ are linearly independent and hence a basis of T_p^*M.*

Proof If $s_1, s_2 \in \mathbb{R}$ are scalars such that

$$s_1\omega_1|_p + s_2\omega_2|_p = 0_{T_p^*M},$$

then

$$0 = s_1\omega_1(\vec{v}_p) + s_2\omega_2(\vec{v}_p) = s_1\langle\vec{v}_p + (\vec{e}_1)_p\rangle_p + s_2\langle\vec{v}_p + (\vec{e}_2)_p\rangle_p = \langle\vec{v}_p, \vec{w}_p\rangle_p$$

for all $\vec{v}_p \in T_pM$, where we write $\vec{w}_p = s_1(\vec{e}_1)_p + s_2(\vec{e}_2)_p$. The vector $\vec{w}_p$ is thus orthogonal to all vectors $\vec{v}_p \in T_pM$. Since $\langle\cdot,\cdot\rangle_p$ is non-degenerate this implies that $\vec{w}_p$ must be the zero vector. This in turn implies that $s_1 = s_2 = 0$, since $(\vec{e}_1)_p, (\vec{e}_2)_p$ are linearly independent. Therefore, $\omega_1|_p, \omega_2|_p$ are linearly independent. Since T_p^*M is two-dimensional, the claim follows. □

Notice that by construction $\vec{e}_3 : M \to M_{3,1}(\mathbb{R})$ is the Gauss map ν of M. In particular the column vector $\mathrm{d}\vec{e}_3(\vec{v}_p) = \mathrm{d}\nu(\vec{v}_p)$ attached at the base point $p \in M$ is tangent to M for all tangent vectors $\vec{v}_p \in T_pM$. This means that there are 1-forms α, β on M so that

$$\mathrm{d}\vec{e}_3(\vec{v}_p) = \alpha(\vec{v}_p)\vec{e}_1 + \beta(\vec{v}_p)\vec{e}_2. \tag{5.6}$$

Since $\omega_1|_p, \omega_2|_p$ are a basis of T_p^*M for all $p \in M$ there are unique functions A_{ij} on M, $1 \leqslant i, j \leqslant 2$ so that

$$\begin{aligned} \alpha|_p &= -A_{11}(p)\omega_1|_p - A_{12}(p)\omega_2|_p, \\ \beta|_p &= -A_{21}(p)\omega_1|_p - A_{22}(p)\omega_2|_p. \end{aligned} \tag{5.7}$$

We now obtain:

Lemma 5.12 *The matrix representation of the shape operator S_p at $p \in M$ with respect to the ordered orthonormal basis $\mathbf{b} = ((\vec{e}_1)_p, (\vec{e}_2)_p)$ is given by*

$$\mathbf{M}(S_p, \mathbf{b}, \mathbf{b}) = -\begin{pmatrix} A_{11}(p) & A_{12}(p) \\ A_{21}(p) & A_{22}(p) \end{pmatrix}.$$

Remark 5.13

(i) Since S_p is self-adjoint and $\mathbf{b}$ an orthonormal basis of T_pM, the matrix $\mathbf{M}(S_p, \mathbf{b}, \mathbf{b})$ is symmetric, hence Lemma 5.12 implies that $A_{21}(p) = A_{12}(p)$.

(ii) For the Gauss curvature at $p \in M$ we thus obtain the formula

$$K(p) = \det \mathbf{M}(S_p, \mathbf{b}, \mathbf{b}) = A_{11}(p)A_{22}(p) - A_{12}(p)^2.$$

Proof of Lemma 5.12 Lemma 3.44 and the definition of the shape operator imply that

$$\mathbf{M}(S_p, \mathbf{b}, \mathbf{b}) = \begin{pmatrix} \mathrm{d}\vec{e}_3((\vec{e}_1)_p) \cdot \vec{e}_1 & \mathrm{d}\vec{e}_3((\vec{e}_1)_p) \cdot \vec{e}_2 \\ \mathrm{d}\vec{e}_3((\vec{e}_2)_p) \cdot \vec{e}_1 & \mathrm{d}\vec{e}_3((\vec{e}_2)_p) \cdot \vec{e}_2 \end{pmatrix}.$$

For the first entry we thus obtain

$$\begin{aligned} \mathrm{d}\vec{e}_3((\vec{e}_1)_p) \cdot \vec{e}_1 &= \left[\alpha((\vec{e}_1)_p)\vec{e}_1 + \beta((\vec{e}_1)_p)\vec{e}_2\right] \cdot \vec{e}_1 = \alpha((\vec{e}_1)_p) \\ &= -A_{11}(p)\omega_1((\vec{e}_1)_p) = -A_{11}(p)\langle(\vec{e}_1)_p, (\vec{e}_1)_p\rangle_p = -A_{11}(p), \end{aligned}$$

where we use $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$, (5.3), (5.6) and (5.7). The calculations for the remaining entries are entirely analogous. □

Combining (5.5) with (5.6) and (5.7) we also have

$$\mathrm{d}\vec{e}_3 = \vec{e}_1\omega_{13} + \vec{e}_2\omega_{23} = -(A_{11}\omega_1 + A_{12}\omega_2)\vec{e}_1 - (A_{12}\omega_1 + A_{22}\omega_2)\vec{e}_2,$$

where we use that $A_{12} = A_{21}$. This implies

$$\begin{aligned} \omega_{13} &= -\omega_{31} = -A_{11}\omega_1 - A_{12}\omega_2, \\ \omega_{23} &= -\omega_{32} = -A_{12}\omega_1 - A_{22}\omega_2. \end{aligned}$$

On the other hand, since $\mathrm{d}^2 = 0$, we obtain

$$\begin{aligned} 0 = \mathrm{d}^2\Psi_3 &= \mathrm{d}\vec{e}_1 \wedge \omega_1 + \vec{e}_1\mathrm{d}\omega_1 + \mathrm{d}\vec{e}_2 \wedge \omega_2 + \vec{e}_2\mathrm{d}\omega_2 \\ &= (\vec{e}_2\omega_{21} + \vec{e}_3\omega_{31}) \wedge \omega_1 + \vec{e}_1\mathrm{d}\omega_1 + (\vec{e}_1\omega_{12} + \vec{e}_3\omega_{32}) \wedge \omega_2 + \vec{e}_2\mathrm{d}\omega_2 \end{aligned}$$

Taking the inner product with $\vec{e}_1$ this simplifies to become

$$0 = \mathrm{d}\omega_1 + \omega_{12} \wedge \omega_2$$

and taking the inner product with $\vec{e}_2$, we obtain

$$0 = \mathrm{d}\omega_2 + \omega_{21} \wedge \omega_1.$$

Writing $\theta := \omega_{21} = -\omega_{12}$ we thus obtain the equations

$$\begin{aligned} \mathrm{d}\omega_1 &= -\omega_2 \wedge \theta, \\ \mathrm{d}\omega_2 &= -\theta \wedge \omega_1. \end{aligned}$$

Taking the exterior derivative of the identity

$$\mathrm{d}\vec{e}_i = \sum_{k=1}^{3} \omega_{ik}\vec{e}_k$$

we conclude likewise that

$$\mathrm{d}\omega_{ij} = -\sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj}.$$

In particular, we have

$$\begin{aligned} \mathrm{d}\theta = \mathrm{d}\omega_{21} &= -\omega_{23} \wedge \omega_{31} = (A_{12}\omega_1 + A_{22}\omega_2) \wedge (A_{11}\omega_1 + A_{12}\omega_2) \\ &= A_{12}^2\omega_1 \wedge \omega_2 + A_{22}A_{11}\omega_2 \wedge \omega_1 = -(A_{11}A_{22} - A_{12}^2)\omega_1 \wedge \omega_2 \\ &= -K\omega_1 \wedge \omega_2. \end{aligned}$$

In summary, we have to so-called *structure equations* of E. Cartan

$$\boxed{\begin{aligned} \mathrm{d}\omega_1 &= -\omega_2 \wedge \theta, \\ \mathrm{d}\omega_2 &= -\theta \wedge \omega_1, \\ \mathrm{d}\theta &= -K\omega_1 \wedge \omega_2. \end{aligned}} \tag{5.8}$$

These equations imply that the Gauss curvature is an intrinsic quantity (i.e. the Theorema Egregium). Indeed, the first two equations of (5.8) imply that ω_1, ω_2 uniquely determine θ. Suppose that $\hat{\theta}$ is another 1-form on M satisfying the first two equations of (5.8). There exist real-valued functions a, b on M so that

$$\hat{\theta} = \theta + a\omega_1 + b\omega_2.$$

The functions a, b exist since $\omega_1|_p, \omega_2|_p$ are basis of T_p^*M for all $p \in M$. By assumption, we have $\mathrm{d}\omega_1 = -\omega_2 \wedge \hat{\theta}$ and hence

$$0 = \mathrm{d}\omega_1 - \mathrm{d}\omega_1 = -\omega_2 \wedge \theta + \omega_2 \wedge \hat{\theta} = -a\omega_1 \wedge \omega_2.$$

Since $\{(\vec{e}_1)_p, (\vec{e}_2)_p\}$ are linearly independent for all $p \in M$ it follows that the alternating bilinear form $(\omega_1 \wedge \omega_2)|_p$ is never the zero form. This implies that a must vanish identically. Arguing with the second equation from (5.8) it follows that b must vanish identically as well, this implies that $\hat{\theta} = \theta$. Using the third equation, one can conclude similarly that K is uniquely determined in terms of $\omega_1, \omega_2, \theta$. Recall that ω_1, ω_2 are intrinsic quantities. Since θ is uniquely determined by ω_1, ω_2, it follows that θ and hence the Gauss curvature K are intrinsic as well.